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Solutions of Nonlinear Integro-Differential Equations Using a Hybrid Adomian Decomposition Method with Modified Bernstein Polynomials

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Abstract

In various scientific disciplines, including mathematics, physics, chemistry, biology, and engineering, numerous challenges are modeled using linear and nonlinear integro-differential equations. Researchers have developed analytical methods to address these issues and to find effective solutions. This study explored the effectiveness of an Adomian decomposition method based on modified Bernstein polynomials to solve nonlinear first-order integro-differential equations. This hybrid method does not require any diminution presumptions or linearization to solve these types of equations, and the arrangement methodology is extremely straightforward with little emphasis on prompting a highly exact solution. This produced an extremely effective strategy among the alternative strategies. The performance of the proposed method was confirmed by comparing the exact and approximate solutions using examples. A comparison of the results shown in numerical tables demonstrates the practical applicability of this method. The computations were performed using the Maple software.

Keywords: Adomian decomposition method (ADM), Bernstein polynomials, Modified Bernstein polynomials, Integro-differential equations, Approximate solutions

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1. Introduction

Recently, modified ADM has been successfully applied to linear and nonlinear problems in various fields, for example, differential equations [1, 2], multi-dimensional time fractional model of the Navier-Stokes equation [3], Van der Pol equation [4], fractional partial differential equations [5], nonlinear integral equations [6], three-dimensional Fredholm integral equations [7], heat and wave equations [8], Newell-Whitehead-Segel Equation [9], time-delay integral equations [10] and others as in [11–25]. Modified Bernstein polynomials were combined with the Adomian decomposition method (ADM) to solve the integro-differential equations. This study focuses on nonlinear integro-differentials of the type.

$$\frac{du}{dx} = f(x) + \int_0^x K\left(t, u(t), u'(t)\right) dt, \tag{1}$$

where $K(t, u(t), u'(t))$ is the kernel function and $f(x)$ is called the source-term.
It should be noted that many methods and techniques have been developed for solving these

types of equations. Examples include the variational iteration [26], Adomian decomposition and Tau Methods [27], homotopy analysis [28], ADM [29], Laplace transform-optimal homotopy asymptotic [30], Mahgoub transform [31], optimal homotopy asymptotic method [32], homotopy perturbation method [33], wavelet-Galerkin method [34], Sumudu and Elzaki integral transforms [35].

In this study, an ADM based on modified Bernstein polynomials was applied to solve nonlinear integro-differential equations. Our aim was to modify the ADM to provide approximate solutions for nonlinear first-order integro-differential equations. This hybrid method finds a solution without discretization or restrictive assumptions and avoids round-off errors. The fundamental advantage of this method is that it can be used specifically for all types of linear and nonlinear differential, integral, and integro-differential equations. First, we briefly present the definitions, properties, and notation of Bernstein polynomials. Herein, we describe our new idea. Furthermore, the convergence and maximum absolute errors of the method were presented. Herein, we discuss some examples of this phenomenon. Additionally, we performed a comparative study with other methods to test the accuracy of the proposed method.

2. Preliminaries

In the following section, we briefly present the basic definitions, most important properties, and notation of the Bernstein polynomials. Bernstein first used these polynomials in 1912 [36, 37].

Definition 1. A linear combination Bernstein based polynomial

$$B_n(x) = \sum_{i=0}^{n} B_{i,n}(x)\beta_i \tag{2}$$

is called the Bernstein polynomials, where $x \in [0, 1]$, β_i are the Bernstein coefficients, and $B_{i,n}(x) = \frac{n!}{i!(n-i)!}x^i(1-x)^{n-i}$.

Definition 2. The nth Bernstein polynomial for $f(x)$ can be written as

$$B_n(f) = \sum_{i=0}^{n} B_{i,n}(x) f\left(\frac{i}{n}\right). \tag{3}$$

We are now giving the most important properties of these polynomials.

1. Non negativity.
2. Symmetry, so $B_{i,n}(x) = B_{n-i,n}(1-x)$.
3. Linearly, so $B_n(\alpha f \mp \beta g) = \alpha B_n(f) \mp \beta B_n(g)$.
4. Each polynomial has only one maximum at $x = \frac{i}{n}$.
5. $\sum_{i=0}^{n} B_{i,n}(x) = B_n(1, x) = 1$.
6. For mathematical convenience, we write $B_{i,n}(x) = 0$ if $i < 0$ or $i > n$.
7. The derivative of the n-th degree was
$$\frac{d}{dx}B_{i,n}(x) = n\left(B_{i-1,n-1}(x) - B_{i,n-1}(x)\right). \tag{4}$$
8. It can be written as
$$B_{i,n}(x) = \left((1-x)B_{i,n-1}(x) + xB_{i-1,n-1}(x)\right), \quad 0 \le i \le n. \tag{5}$$

Remark 1. The $2k$-th order derivative $f^{(2k)}$ given by

$$B_n^f(x) = f(x) + \sum_{a=2}^{2k-1} \frac{f^{(a)}(x)}{a!n^a} T_{n,a}(x) + O\left(\frac{1}{n^k}\right), \tag{6}$$

where

$$T_{n,a}(x) = \sum_{k} (k - nx)^a \binom{n}{k} x^k (1-x)^{n-k}. \tag{7}$$

3. Description of the Method

This section describes the application of the modified ADM with a Bernstein polynomial for solving the nonlinear first-order integro-differential equations. We will rewrite Eq. (1) as follows:

$$\mathcal{L}u = f(x) + \int_0^x N(u)dt, \tag{8}$$

where $\mathcal{L}u = \frac{du}{dx}$ and $N(u)$ is a nonlinear term.

To solve this problem, we take $\mathcal{L}^{-1}$ such that $\mathcal{L}^{-1}(.) = \int_0^x .dx$ to both sides

$$u(x) - u(0) = g(x) + \mathcal{L}^{-1}\int_0^x N(u)dt, \tag{9}$$

where $g(x) = \int_0^x f(t)dx$.

The Adomian polynomial $N(u)$ is as follows:

$$N(u) = \int_{n=0}^{\infty} A_n, \tag{10}$$

where $A_n = \frac{1}{n!}\frac{d^n}{d\gamma^n}\left[N\left(\sum_{i=0}^{\infty}\gamma^i u_i\right)\right]$, $n = 0, 1, 2, \ldots$.

Using Eqs. (3), (4), and (6), we obtain the following modified Bernstein series:

$$f(x)=\sum_{i=0}^{n}\binom{n}{i}x^{i}(1-x)^{n-i}g\left(\frac{i}{n}\right)-\sum_{a=2}^{2k-1}\frac{\left(\frac{d^{a}}{dx^{a}}\right)B_{i,n}(x)}{a!n^{a}}T_{n,a}(x). \tag{11}$$

The solution $u(x)$ is given by

$$u(x)=\int_{j=0}^{\infty}u_j(x). \tag{12}$$

Substituting Eqs. (10), (11), and (12) into Eq. (9), we have

$$\sum_{j=0}^{\infty}u_j(x)=u(0)+\sum_{i=0}^{n}\binom{n}{i}x^{i}(1-x)^{n-i}g\left(\frac{i}{n}\right)-\sum_{a=2}^{2k-1}\frac{\left(\frac{d^{a}}{dx^{a}}\right)B_{i,n}(x)}{a!n^{a}}T_{n,a}(x)+\mathcal{L}^{-1}\int_0^x A_n dt. \tag{13}$$

Therefore, the solutions of Eq. (1) are given by

$$\begin{aligned}
u_0(x)&=u(0)+\mathcal{L}^{-1}\sum_{i=0}^{n}\binom{n}{i}x^{i}(1-x)^{n-i}g\left(\frac{i}{n}\right)-\sum_{a=2}^{2k-1}\frac{\left(\frac{d^{a}}{dx^{a}}\right)B_{i,n}(x)}{a!n^{a}}T_{n,a}(x),\\
u_1(x)&=\mathcal{L}^{-1}\int_0^x A_0 dt,\\
u_2(x)&=\mathcal{L}^{-1}\int_0^x A_1 dt,\\
&\vdots\\
u_{n+1}(x)&=\mathcal{L}^{-1}\int_0^x A_n dt,\ n=0,\ 1,\ 2,\ \ldots.
\end{aligned} \tag{14}$$

4. Convergence of the Method

This section presents the convergence of the proposed method for solving nonlinear integro-differential equations. Furthermore, the maximum absolute error is determined.

Theorem 1. The series $u_n(x)=\sum_{j=0}^{\infty}u_j(x)$ of Eq. (1) converges if $\alpha\in[0,1)$ and $\|u_{n+1}(x)\|=\alpha\|u_n(x)\|$ such that $\|u_0(x)\|<\infty$.

Proof. Let $S_n=u_1(x)+u_2(x)+...+u_n(x)$. For $n\le m$, we are going to prove that S_n is a Cauchy sequence.

$$\|S_m-S_n\|=\left\|\int_{i=0}^{m}u_i(x)-\int_{i=0}^{n}u_i(x)\right\|=\int_{i=n+1}^{m}u_i(x).$$

We have the following sequence

$$\begin{aligned}
\|S_m-S_n\|&=\big\|(S_{n+1}-S_n)+(S_{n+2}-S_{n+1})+(S_{n+3}-S_{n+2})\\
&\quad+\cdots+(S_m-S_{m-1})\big\|\\
&\le\|(S_{n+1}-S_n)\|+\|(S_{n+2}-S_{n+1})\|\\
&\quad+\|(S_{n+3}-S_{n+2})\|+\cdots+\|(S_m-S_{m-1})\|\\
&\le\alpha^{n+1}\|(u_0)\|+\alpha^{n+2}\|(u_0)\|+\alpha^{n+3}\|(u_0)\|\\
&\quad+\cdots+\alpha^{m}\|(u_0)\|\\
&\le\left(\alpha^{n+1}+\alpha^{n+2}+\alpha^{n+3}+\cdots+\alpha^{m}\right)\|(u_0)\|\\
&\le\alpha^{n+1}\left(1+\alpha+\alpha^2+\alpha^3\cdots+\alpha^{m-n-1}\right)\|(u_0)\|\\
&\le\alpha^{n+1}\left(\frac{1-\alpha^{m-n}}{1-\alpha}\right)\|(u_0)\|.
\end{aligned}$$

Since $\alpha\in[0,1)$ then $1-\alpha^{m-n}<1$.

This in turn gives

$$\|S_m-S_n\|\le\left(\frac{\alpha^{n+1}}{1-\alpha}\right)\|(u_0)\|.$$

But we know that $\|u_0(x)\|<\infty$, yields

$$\|S_m-S_n\|\to 0 \text{ as } n\to 0.$$

Therefore, Theorem 1 is proven. □

Theorem 2. The maximum absolute error of $u_n(x)=\sum_{j=0}^{\infty}u_j(x)$ is

$$\max_{\forall x\in J}\left|u(x)-\sum_{i=0}^{n}u_i(x)\right|\le\left(\frac{\alpha^{n+1}}{1-\alpha}\right)\max_{\forall x\in J}\|(u_0)\|.$$

Proof. If $m\to\infty$ then $S_m\to u(x)$, yields

$$\|u(x)-S_n\|\le\left(\frac{\alpha^{n+1}}{1-\alpha}\right)\|(u_0)\|.$$

Therefore, we have

$$\max_{\forall x\in J}\left|u(x)-\int_{i=0}^{n}u_i(x)\right|\le\left(\frac{\alpha^{n+1}}{1-\alpha}\right)\max_{\forall x\in J}\|(u_0)\|.$$

Therefore, Theorem 2 is proven. □

5. Examples

The main objective here is to solve some examples of nonlinear first-order integro-differential equations using the new modified method to demonstrate its accuracy.

Example 1. Consider the nonlinear integro-differential equation as follows:

$$\frac{du}{dx} = -1 + \int_0^x u^2(t)dt,\ x \in [0,\ 1], \tag{15}$$

with boundary condition $u(0) = 0$ and exact solution $u(x) = -x$.

Introducing Eq. (10) into Eq. (15) and taking $\mathcal{L}^{-1}$ to both sides, yields

$$u(x) - u(0) = \mathcal{L}^{-1}\left(-1 + \int_0^x \sum_{n=0}^{\infty} A_n(t)dt\right). \tag{16}$$

Combining Eq. (12) and Eq. (16), we have

$$\sum_{j=0}^{\infty} u_j(x) = \mathcal{L}^{-1}\left(-1 + \int_0^x \sum_{n=0}^{\infty} A_n(t)dt\right). \tag{17}$$

As suggested in Eqs. (11)-(14), when $m = 6$ and $k = 2$, the following solutions were obtained.

$$\begin{aligned}
u_0(x) &= \mathcal{L}^{-1}\sum_{i=0}^{6}\binom{6}{i}x^i(1-x)^{6-i}f\left(\frac{i}{6}\right) - \sum_{a=2}^{3}\frac{\left(\frac{d^a}{dx^a}\right)B_{i,6}(x)}{a!6^a}T_{6,a}(x)\\
&= -x(1-x)^5 - 5x^2(1-x)^4 - 10x^3(1-x)^3 - 10x^4(1-x)^2\\
&\quad - 5x^5(1-x) - x^6 = -x,\\
u_1(x) &= \int_0^x\int_0^x A_0dtdx = \int_0^x\int_0^x {u_0}^2(t)dtdx = \frac{1}{12}x^4,\\
u_2(x) &= \int_0^x\int_0^x A_1dtdx = \int_0^x\int_0^x 2u_0(t)u_1(t)dtdx = -\frac{1}{252}x^7,\\
u_3(x) &= \int_0^x\int_0^x A_2dtdx = \int_0^x\int_0^x 2u_0(t)u_2(t) + {u_1}^2(t)dtdx\\
&= \frac{1}{6048}x^{10},\\
u_4(x) &= \int_0^x\int_0^x A_3dtdx = \int_0^x\int_0^x 2u_0(t)u_3(t) + 2u(t)u_2(t)dtdx\\
&= -\frac{1}{157248}x^{13},
\end{aligned}$$

and so on.

The solution of Eq. (15) becomes

$$\begin{aligned}
u_m(x) &= \int_{j=0}^{m} u_i = u_0(x) + u_1(x) + u_2(x) + \ldots\\
&= -x + \frac{1}{12}x^4 - \frac{1}{252}x^7 + \frac{1}{6048}x^{10} - \frac{1}{157248}x^{13} + \ldots.
\end{aligned} \tag{18}$$

This converges to the exact solution $u(x) = -x$. Table 1 shows a comparison between the first fourth-order approximate solutions $u_4(x)$ of Example 1 and the solutions in [29, 33, 34]. Figure 1 presents a numerical comparison of the proposed solution, $u_4(x)$ and the exact solution. Figure 2 shows the absolute errors for Example 1.

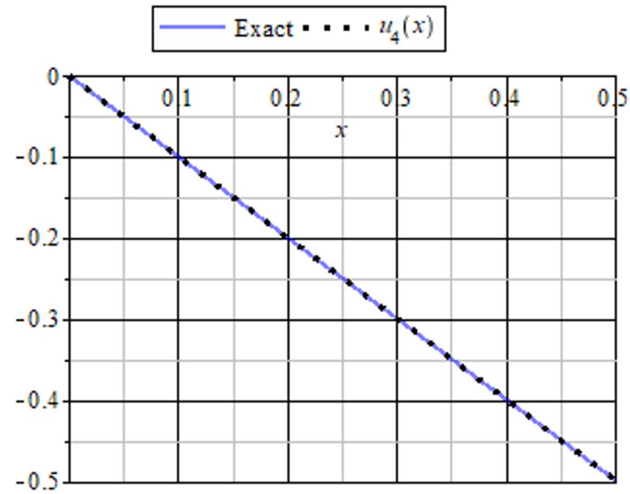

Figure 1. Numerical comparison between our solution $u_4(x)$ and the exact solution for Example 1.

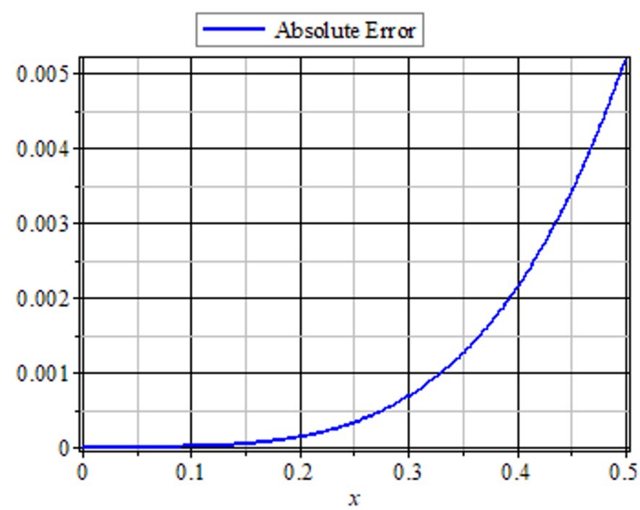

Figure 2. Absolute error for Example 1.

Example 2. Consider the nonlinear integro-differential equation as follows:

$$\frac{du}{dx} = 1 + \int_0^x e^{-t}u^2(t)dt, \tag{19}$$

with boundary condition $u(0) = 1$ and exact solution $u(x) = e^x$.

Using Eq. (10) in Eq. (19) and $\mathcal{L}^{-1}$ on both sides, we obtain

$$\sum_{j=0}^{\infty} u_j(x) = u(0) + \mathcal{L}^{-1}\left(1 + \int_0^x \sum_{n=0}^{\infty} A_n(t)dt\right).$$

Table 1. Comparison between our first fourth-order approximate solutions $u_4(x)$ of Example 1 and the solutions in other studies

x	Exact	$u_4(x)$	ADM [29]	WGM [33]	HPM [34]
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0312	- 0.0312	-0.03119992103	- 0.0311999	-0.0312	-0.03120
0.0625	- 0.0625	-0.06249872844	- 0.0624987	-0.0625	-0.06250
0.0938	- 0.0938	-0.09379354920	- 0.0937935	-0.0937	-0. 09380
0.1250	- 0.1250	-0.3978731510	- 0.1249800	-0.1250	-0. 12498
0.1562	- 0.1562	-0.1561504020	- 0.1561500	-0.1562	-0. 15615
0.1875	- 0.1875	-0.1873970355	- 0.1873970	-0.1874	-0.18740
0.2188	- 0.2188	-0.2186091064	- 0.2186090	- 0.2186	-0.21861
0.2500	- 0.2500	-0.2496747212	- 0.2496750	- 0.2497	-0.24968
0.2812	- 0.2812	-0.2806795005	- 0.2806800	- 0.2807	-0.28068
0.3125	- 0.3125	-0.3117064248	- 0.3117060	- 0.3117	-0.31171

As suggested in Eqs. (11)-(14) when $m = 6$ and $k = 2$, yields

$$
\begin{aligned}
u_0(x) &= u(0) + \mathcal{L}^{-1} \sum_{i=0}^{6} \binom{6}{i} x^i (1-x)^{6-i} f\left(\frac{i}{6}\right) \\
&\quad - \sum_{a=2}^{3} \frac{\left(\frac{d^a}{dx^a}\right) B_{i,6}(x)}{a!6^a} T_{6,a}(x) \\
&= 1 + x(1-x)^5 + 5x^2(1-x)^4 + 10x^3(1-x)^3 \\
&\quad + 10x^4(1-x)^2 + 5x^5(1-x) + x^6 \\
&= 1 + x, \\
u_1(x) &= \int_0^x \int_0^x A_0 dtdx \\
&= \int_0^x \int_0^x e^{-t} u_0^2(t) dtdx \\
&= -11 + x^2 e^{-x} + 6xe^{-x} + 11e^{-x} + 5x, \\
u_2(x) &= \int_0^x \int_0^x A_1 dtdx \\
&= \int_0^x \int_0^x 2e^{-t} u_0(t) u_1(t) dtdx \\
&= \frac{-139}{4} + \frac{39}{4} x + 14e^{-x} + 5e^{-2x} x^2 + \frac{71}{4} e^{-2x} x \\
&\quad + \frac{83}{4} e^{-2x} + 28xe^{-x} + \frac{1}{2} x^3 e^{-2x} + 10x^2 e^{-x}, \\
u_3(x) &= \int_0^x \int_0^x A_2 dtdx \\
&= \frac{1}{972}(15267xe^{3x} + 43254x^2 e^{2x} + 7290x^3 e^x \\
&\quad + 216x^4 - 676073 + 17496xe^{2x} + 49572x^2 e^x \\
&\quad + 3060x^3 - 1458e^{2x} + 103275xe^x + 17010x^2 \\
&\quad + 26487e^x + 43212x + 42578)e^{-3x},
\end{aligned}
$$

$$
\begin{aligned}
u_4(x) &= \int_0^x \int_0^x A_3 dtdx \\
&= \frac{1}{248832}(5536734xe^{4x} + 32077824x^2 e^{3x} \\
&\quad + 12970368x^3 e^{2x} + 1105920x^4 e^x + 22464x^5 \\
&\quad - 28049407e^{4x} - 38330368xe^{3x} + 53374464x^2 e^{2x} \\
&\quad + 12128256x^3 e^x + 409824x^4 + 14800896e^{3x} \\
&\quad + 47138112xe^{2x} + 49489920x^2 e^x + 3090096x^3 \\
&\quad - 46453824e^{2x} + 81899520xe^x + 11923584x^2 \\
&\quad + 41003008e^x + 23455582x + 18699327)e^{-4x},
\end{aligned}
$$

and so on. Therefore, the solution of Eq. (19) is expressed as

$$
\begin{aligned}
u_m(x) &= \sum_{j=0}^{m} u_i \\
&= u_0(x) + u_1(x) + u_2(x) + \ldots \\
&= 1 + x - 11 + x^2 e^{-x} + 6xe^{-x} + 11e^{-x} + 5x + \ldots. \quad (20)
\end{aligned}
$$

This converges to the exact solution, $u(x) = e^x$. Table 2 and Figures 3 and 4 present a numerical comparison between our solution, $u_4(x)$ and the exact solution for Example 2.

Example 3. Consider the following nonlinear integro-differential equation:

$$
\frac{du}{dx} = \frac{3}{2} e^x - \frac{1}{2} e^{-3x} + \int_0^x e^{x-t} u^3(t) dt, \; u(0) = 1. \quad (21)
$$

Applying the Adomian polynomial and $\mathcal{L}^{-1}$ to both sides,

Table 2. Numerical comparison between our solution $u_4(x)$ and the exact solution for Example 2

x	Exact	$u_4(x)$	Absolute error
0	1.000000000	1.000000000	0.00000000000
0.1	1.105170918	1.105170947	2.748263881 e^{-8}
0.2	1.221402758	1.221402795	3.470977888 e^{-8}
0.3	1.349858808	1.349858866	5.617735570 e^{-8}
0.4	1.491824698	1.491824726	2.548139800 e^{-8}
0.5	1.648721271	1.648721069	2.030066670 e^{-7}
0.6	1.822118800	1.822117695	1.105495050 e^{-6}
0.7	2.013752707	2.013747690	5.019781840 e^{-6}
0.8	2.225540928	2.225522980	1.795120140 e^{-5}
0.9	2.459603111	2.459549162	5.394816930 e^{-5}
1.0	2.718281828	2.718138102	1.437281820 e^{-4}

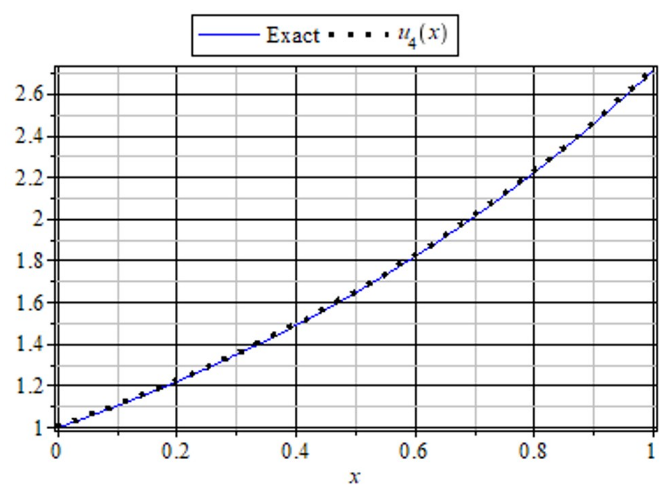

Figure 3. Numerical comparison between our solution $u_4(x)$ and exact solution for Example 2.

we have

$$\sum_{j=0}^{\infty} u_j(x) = u(0) + \mathcal{L}^{-1}\left(\frac{3}{2}e^x - \frac{1}{2}e^{-3x} + \int_0^x \sum_{n=0}^{\infty} A_n(t)dt\right).$$

This formula gives the following series solution:

$$\begin{aligned} u_m(x) &= \sum_{j=0}^{m} u_i = u_0(x) + u_1(x) + u_2(x) + \dots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots . \end{aligned} \quad (22)$$

This converges to the exact solution, $u(x) = e^x$. Table 3 and Figure 5 present a numerical comparison between the proposed solution and the exact solution for Example 3.

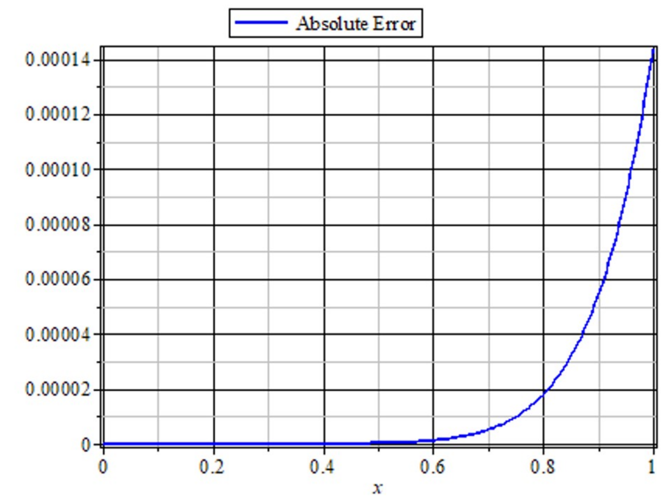

Figure 4. Absolute error for Example 2.

Table 3. Numerical comparison between our solution and the exact solution for Example 3

x	Exact	$u_{App}(x)$	Absolute error
0	1.000000000	1.000000000	0.00000000000
0.1	1.105170918	1.105170917	1.33329967 e^{-9}
0.2	1.221402758	1.221402667	9.1333663 e^{-8}
0.3	1.349858808	1.349857750	1.05800000 e^{-6}
0.4	1.491824698	1.491818667	6.03132967 e^{-6}
0.5	1.648721271	1.648697917	2.33543363 e^{-5}
0.6	1.822118800	1.822048000	7.08000000 e^{-5}
0.7	2.013752707	2.013571417	1.81290327 e^{-4}
0.8	2.225540928	2.225130667	4.10261323 e^{-4}
0.9	2.459603111	2.458758250	8.44861000 e^{-4}
1.0	2.718281828	2.716666667	1.615161297 e^{-3}

6. Conclusion

We constructed an ADM based on modified Bernstein polynomials to solve the nonlinear first-order integro-differential equations. The performance of this method was validated by comparing the exact and approximate solutions for some examples. The results confirmed that this hybrid method can compete with other efficient methods for solving these types of equations. This method did not require any diminutive presumptions to solve the nonlinear integro-differential equations and produced an extremely effective strategy among the alternate strategies. This hybrid method is suitable for solving nonlinear problems.

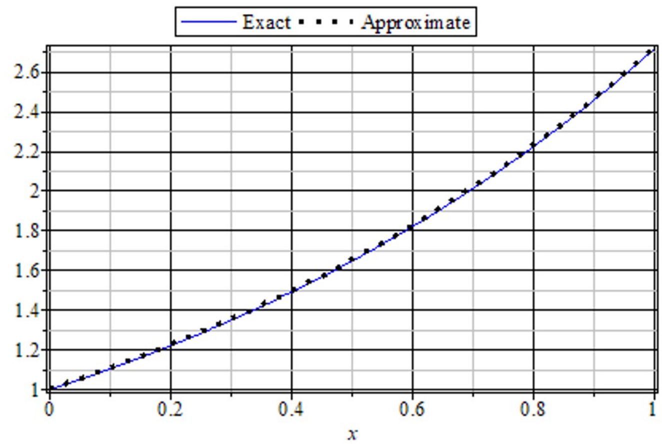

Figure 5. Numerical comparison between our solution and the exact solution for Example 3.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

References

[1] A. F. Qasim and E. S. Al-Rawi, "Adomian decomposition method with modified Bernstein polynomials for solving ordinary and partial differential equations," *Journal of Applied Mathematics*, vol. 2018, article no. 1803107, 2018. https://doi.org/10.1155/2018/1803107

[2] B. Zhang and J. Lu, "Exact solutions of homogeneous partial differential equation by a new adomian decomposition method," *Procedia Environmental Sciences*, vol. 11(Part A), pp. 440-446, 2011. https://doi.org/10.1016/j.proenv.2011.12.070

[3] S. Mahmood, R. Shah, H. Khan, and M. Arif, "Laplace adomian decomposition method for multi dimensional time fractional model of Navier-Stokes equation," *Symmetry*, vol. 11, no. 2, article no. 149, 2019. https://doi.org/10.3390/sym11020149

[4] P. V. Ramana and B. R. Prasad, "Modified Adomian decomposition method for Van der Pol equations," *International Journal of Non-Linear Mechanics*, vol. 65, pp. 121-132, 2014. https://doi.org/10.1016/j.ijnonlinmec.2014.03.006

[5] R. Shah, H. Khan, P. Kumam, and M. Arif, "An analytical technique to solve the system of nonlinear fractional partial differential equations," *Mathematics*, vol. 7, no. 6, article no. 505, 2019. https://doi.org/10.3390/math7060505

[6] M. Almousa, "Adomian decomposition method with modified Bernstein polynomials for solving nonlinear Fredholm and Volterra integral equations," *Mathematics and Statistics*, vol. 8, no. 3, pp. 278-285, 2020. https://doi.org/10.13189/ms.2020.080305

[7] M. Almousa, "The solutions of three dimensional Fredholm integral equations using Adomian decomposition method," *AIP Conference Proceedings*, vol. 1739, article no. 020053, 2016. https://doi.org/10.1063/1.4952533

[8] M. Almousa, I. Al-zuhairi, Y. Al-Qudah, and H. Qoqazeh, "Solution of heat and wave equations using Mahgoub Adomian decomposition method," *International Advance Journal of Engineering Research*, vol. 3, no. 1, pp. 7-11, 2020.

[9] M. Almousa, I. Al-Zuhairi, Y. Al-Qudah, and H. Qoqazeh, "Mahgoub Adomian decomposition method for solving Newell-Whitehead-Segel equation," *International Journal of Mathematics and Statistics Invention*, vol. 8, no. 1, pp. 22-24, 2020. https://ijmsi.org/Papers/Volume.8.Issue.1/D08012224.pdf

[10] M. Almousa and O. N. Ogilat, "A new approach for solution of linear time-delay integral equations via Z-decommission method," *Far East Journal of Mathematical Sciences*, vol. 102, no. 11, pp. 2551-2561, 2017. https://doi.org/10.17654/MS102112551

[11] L. Song and W. Wang, "A new improved Adomian decomposition method and its application to fractional differential equations," *Applied Mathematical Modelling*, vol. 37, no. 3, pp. 1590-1598, 2013. https://doi.org/10.1016/j.apm.2012.03.016

[12] A. M. Wazwaz and S. M. El-Sayed, "A new modification of the Adomian decomposition method for linear and nonlinear operators," *Applied Mathematics and Computation*, vol. 122, no. 3, pp. 393-405, 2001. https://doi.org/10.1016/S0096-3003(00)00060-6

[13] Y. Liu, "Adomian decomposition method with orthogonal polynomials: legendre polynomials," *Mathematical and Computer Modelling*, vol. 49, no. 5-6, pp. 1268-1273, 2009. https://doi.org/10.1016/j.mcm.2008.06.020

[14] Y. Daoud and A. A. Khidir, "Modified Adomian decomposition method for solving the problem of boundary layer convective heat transfer," *Propulsion and Power Research*, vol. 7, no. 3, pp. 231-237, 2018. https://doi.org/10.1016/j.jppr.2018.05.005

[15] M. Almousa and A. I. M. Ismail, "Equivalence between homotopy perturbation method and Adomian decomposition method in solving two dimensional linear Fredholm integral equations," *AIP Conference Proceedings*, vol. 1635, no. 1, pp. 45-51, 2014. https://doi.org/10.1063/1.4903561

[16] E. H. Aly, A. Ebaid, and R. Rach, "Advances in the Adomian decomposition method for solving two-point nonlinear boundary value problems with Neumann boundary conditions," *Computers & Mathematics with Applications*, vol. 63, no. 6, pp. 1056-1065, 2012. https://doi.org/10.1016/j.camwa.2011.12.010

[17] K. O. Kareem, M. Olayiwola, O. Asimiyu, Y. Akeem, K. Adedokun, and I. Alaje, "On the solution of Volterra integro-differential equations using a modified Adomian decomposition method," *Jambura Journal of Mathematics*, vol. 5, no. 2, pp. 265-277, 2023. https://doi.org/10.34312/jjom.v5i2.19029

[18] B. Manimegalai, R. Swaminathan, M. E. Lyons, and L. Rajendran, "Homogeneous catalysis of electrochemical reactions: the steady-state current and turnover frequency," *The Journal of Physical Chemistry C*, vol. 127, no. 24, pp. 11517-11525, 2023. https://doi.org/10.1021/acs.jpcc.3c01507

[19] R. U. Rani, L. Rajendran, and M. Abukhaled, "Theoretical analysis of homogeneous catalysis of electrochemical reactions: steady-state current–potential," *Reaction Kinetics, Mechanisms and Catalysis*, vol. 136, pp. 1229-1242, 2023. https://doi.org/10.1007/s11144-023-02407-x

[20] J. Saranya, P. Jeyabarathi, L. Rajendran, and M. E. G. Lyons, "Modelling and simulation of microbial activity in stratified and homogeneous biofilms: hyperbolic function method," *International Journal of Electrochemical Science*, vol. 17, no. 12, article no. 221277, 2022. https://doi.org/10.20964/2022.12.68

[21] A. I. Alaje, M. O. Olayiwola, M. O. Ogunniran, J. A. Adedeji, and K. A. Adedokun, "Approximate analytical methods for the solution of fractional order integro-differential equations," *Nigerian Journal of Mathematics and Applications*, vol. 31, pp. 175-190, 2021.

[22] M. O. Olayiwola and K. Kareem, "A new decomposition method for integro-differential equations," *Cumhuriyet Science Journal*, vol. 43, no. 2, pp. 283-288, 2022. https://doi.org/10.17776/csj.986019

[23] M. Turkyilmazoglu, "Nonlinear problems via a convergence accelerated decomposition method of Adomian," *Computer Modeling in Engineering & Sciences*, vol. 127, no. 1, pp. 1-22, 2021. https://doi.org/10.32604/cmes.2021.012595

[24] M. Turkyilmazoglu, "High-order nonlinear Volterra–Fredholm-Hammerstein integro-differential equations and their effective computation," *Applied Mathematics and Computation*, vol. 247, pp. 410-416, 2014. https://doi.org/10.1016/j.amc.2014.08.074

[25] M. Turkyilmazoglu, "A reliable convergent Adomian decomposition method for heat transfer through extended surfaces," *International Journal of Numerical Methods for Heat & Fluid Flow*, vol. 28, no. 11, pp. 2551-2566, 2018. https://doi.org/10.1108/HFF-01-2018-0003

[26] S. Q. Wang and J. H. He, "Variational iteration method for solving integro-differential equations," *Physics Letters A*, vol. 367, no. 3, pp. 188-191, 2007. https://doi.org/10.1016/j.physleta.2007.02.049

[27] N. Bildik and M. Inc, "A comparison between Adomian decomposition and Tau methods," *Abstract and Applied Analysis*, vol. 2013, article no. 621019, 2013. https://doi.org/10.1155/2013/621019

[28] Z. Abbas, S. Vahdati, F. Ismail, and A. K. Dizicheh, "Application of homotopy analysis method for linear integro-differential equations," *International Mathematical Forum*, vol. 5, no. 5, pp. 237-249, 2010.

[29] S. M. El-Sayed and M. R. Abdel-Aziz, "A comparison of Adomian's decomposition method and wavelet-Galerkin method for solving integro-differential equations," *Applied Mathematics and Computation*, vol. 136, no. 1, pp. 151-159, 2003. https://doi.org/10.1016/S0096-3003(02)00024-3

[30] M. Almousa, A. Al-Hammouri, S. Saidat, S. Alsaadi, and G. Banihani, "A hybrid Laplace transform-optimal homo-

topy asymptotic method (LT-OHAM) for solving integro-differential equations of the second kind," *International Journal of Analysis and Applications*, vol. 22, pp. 17-17, 2024. https://doi.org/10.28924/2291-8639-22-2024-17

[31] M. Almousa, "Solution of second order initial-boundary value problems of partial integro-differential equations by using a new transform: Mahgoub transform," *European Journal of Advances in Engineering and Technology*, vol. 5, no. 10, pp. 802-805, 2018. https://doi.org/10.5281/zenodo.10726977

[32] H. Temimi, A. R. Ansari, and A. M. Siddiqui, "An approximate solution for the static beam problem and nonlinear integro-differential equations," *Computers & Mathematics with Applications*, vol. 62, no. 8, pp. 3132-3139, 2011. https://doi.org/10.1016/j.camwa.2011.08.026

[33] M. Ghasemi and E. Babolian, "Numerical solutions of the nonlinear integro-differential equations: Wavelet-Galerkin method and homotopy perturbation method," *Applied Mathematics and Computation*, vol. 188, no. 1, pp. 450-455, 2007. https://doi.org/10.1016/j.amc.2006.10.001

[34] A. Avudainayagam and C. Vani, "Wavelet–Galerkin method for integro–differential equations," *Applied Numerical Mathematics*, vol. 32, no. 3, pp. 247-254, 2000. https://doi.org/10.1016/S0168-9274(99)00026-4

[35] M. Almousa, A. Al-Hammouri, Q. A. Almomani, M. H. Alomari, and M. M. Alzubaidi, "Sumudu and Elzaki integral transforms for solving systems of integral and ordinary differential equations," *Advances in Differential Equations and Control Processes*, vol. 31, no. 1, pp. 43-60, 2024. https://doi.org/10.17654/0974324324004

[36] G. G. Lorentz, *Bernstein Polynomials*, 2nd ed. New York, NY: Chelsea Publishing, 1986.

[37] J. Cichon and Z. Golebiewski, "On Bernoulli sums and Bernstein polynomials," in *Discrete Mathematics & Theoretical Computer Science Proceedings (AofA'12)*, 2012. pp. 179-190. https://doi.org/10.46298/dmtcs.2993

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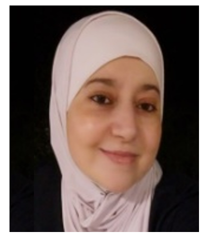

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