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Prime Ideal Theorem for Fuzzy Lattices

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Abstract

In this study, the fuzzy prime ideal theorem is established by adopting an unusual technique of quasi-coincident (overlapping) fuzzy sets. Subsequently, the intriguing applications of the fuzzy prime ideal theorem verify its substantial importance in fuzzy algebra. Every proper fuzzy ideal in a distributive lattice L is the intersection of the fuzzy prime ideals of L. The author also proved that the existence of a fuzzy prime ideal in a sublattice of L ensures the existence of a fuzzy prime ideal in L. Moreover, the classical prime ideal theorem for lattices is a corollary of the fuzzy prime ideal theorem.

Keywords: Fuzzy lattice distributive, Lattice prime ideal, Quasi coincident fuzzy sets, Fuzzy point

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1. Introduction

In 1971, Rosenfeld [1] applied the notion of fuzzy sets to algebra and formulated the concepts of the fuzzy subgroupoid and fuzzy subgroups of a group in his seminal paper. In 1981, Wu [2] defined the notion of a normal fuzzy subgroup and studied the related concepts. Liu [3] conducted this study and defined the concept of a fuzzy subring. Subsequently, researchers worldwide have applied the notion of fuzzy sets to their respective fields to define various fuzzy algebraic concepts [4–8]. Researchers have also explored the theory of L fuzzy sets introduced by Goguen [9]. Yuan and Wu [10] applied the notion of fuzzy sets to lattice theory and initiated the study of fuzzy lattices by defining the concepts of fuzzy sublattice and fuzzy ideals in a lattice.

The fuzzy lattice theory was systematically developed by Ajmal and Thomas [11]. They introduced concepts such as fuzzy dual ideals, fuzzy convex sublattices, fuzzy sublattices, and fuzzy ideals (dual ideals) generated by a fuzzy set. The unique representation theorem for convex sublattices was extended to fuzzy settings [11]. In [12, 13], the authors studied fuzzy congruences and fuzzy ideals in a lattice and introduced operations for fuzzy sets in a lattice. They also prove that a lattice is distributive iff the lattice of its fuzzy ideals (dual ideals) is distributive. The notions of the fuzzy prime ideal and fuzzy dual-prime ideal were also introduced and studied. by Ajmal and Thomas [11–13], Thereafter, several researchers continued to work in fuzzy lattices [14, 15].

The prime ideal theorem is one of the most important results of this theory for the distributive lattices [16]. In lattice theory, the prime ideal theorem for distributive lattices is equivalent to the maximal ideal theorem for Boolean algebra. The equivalence of the existence of a maximal ideal in a distributive lattice with 1 to the axiom of choice is established, however, the prime ideal theorem for distributive lattices does not share this equivalence. The axiom of choice implies the prime ideal theorem but not vice versa.

In this study, we established a fuzzy prime ideal theorem. In [17, 18], the authors have proposed a fuzzy version of this well-known prime ideal theorem from classical lattice theory. In [18], the proof of the fuzzy prime ideal theorem heavily relied on the classical prime ideal theorem of the lattice theory (based on Stone). In fact, in [18], the authors at each level "α", have used the prime ideal theorem (where α is arbitrary but fixed in [0, 1]) to establish their result. However, in this study, we have formulated the fuzzy version of the prime ideal theorem such that, in its proof, we use Zorn's lemma only. The classical prime ideal theorem follows as a simple corollary of our results. In addition, in the proof of the fuzzy prime ideal theorem in [18], the fact that the subsets I = μ_α and J = ν_α are ideal and filtered, respectively, does not follow from their work, as these are not α-level subsets, but strong level subsets. However, their claim is valid, and the proof of the result remains correct. This was a significant attempt in this direction. The authors of [17, 18] did not show that the classical version follows the fuzzy prime ideal theorem. Considering the above discussion, our fuzzy prime ideal theorem is stronger than the classical prime ideal theorem. However, the fuzzy prime ideal theorem in [18] is equivalent to the classical prime ideal theorem.

In this study, we state and establish the fuzzy prime ideal theorem and prove related results that authenticate the fact that the fuzzy prime ideal theorem can be further used to advance the theory of fuzzy lattices. The prime ideal theorem in classical lattice algebra follows a simple corollary of this fuzzy version. In this study, the fuzzy prime ideal theorem was obtained by adopting an unusual and unique technique of quasi-coincident (overlapping) fuzzy sets and disquasi-coincident (non-overlapping) fuzzy sets. These notions were first employed by Pu and Liu [19] in their pioneering study on fuzzy topologies. In two quasi-coincident fuzzy sets, they replaced one of the fuzzy sets by a fuzzy point and thus defined a relationship of quasi-coincidence between a fuzzy point and a fuzzy set and this replaced the notion of "belonging to" in classical set theory. This was instrumental in the formation of a quasi-neighborhood system, in which laid the foundation for the successful development of the fuzzy topological space theory. In [20], the authors used this concept and introduced the notion of overlapping families of fuzzy sets and the order of a family of fuzzy sets.

2. Preliminaries

In this section, we present a few basic definitions and results that were subsequently used. A fuzzy set μ in set S is defined as the mapping from S to closed interval $[0, 1]$. If μ and η are fuzzy sets in S, then μ is said to be contained in η if $\mu(x) \leq \eta(x)$ $\forall$ $x \in S$, denoted by $\mu \subseteq \eta$. The arbitrary union $\bigcup_{i\in I}(\mu_i)$ and intersection $\bigcap_{i\in I}(\mu_i)$ of a family of fuzzy sets $(\mu_i)_{i\in I}$ in set S are defined as fuzzy sets in S given by

$$\left(\bigcup \mu_i\right)(x) = \sup\{\mu_i(x) : i \in I\},$$

and

$$\left(\bigcap \mu_i\right)(x) = \inf\{\mu_i(x) : i \in I\}.$$

Throughout this study, L denotes a lattice, and $\mathcal{F}(L)$ denotes the set of all fuzzy sets in L. If $\mu \in \mathcal{F}(L)$, the complement of μ, denoted by μ' is a fuzzy set in L defined by $\mu'(x) = 1 - \mu(x)$ $\forall$ $x \in L$.

The notions of level and strong level subsets are crucial in establishing numerous results and characterizations of fuzzy algebraic structures. If μ is a fuzzy set in set S, then for $t \in [0, 1]$, a level subset μ_t and strong level subset $\mu_t^>$ are defined as

$$\mu_t = \{x \in S/\mu(x) \geq t\},$$
$$\mu_i^> = \{x \in S/\mu(x) > t\},$$

respectively. Yuan and Wu [10] introduced the notion of a fuzzy sublattice in a lattice as follows:

Definition 2.1 [10]**.** A fuzzy set μ in L is called a fuzzy sublattice of L if

$$\mu(x + y) \geq \min\{\mu(x), \mu(y)\} < \quad \text{and}$$
$$\mu(xy) \geq \min\{\mu(x), \mu(y)\} \quad \forall\, x, y \in L.$$

Let $\mathcal{L}(L)$ denote a set of fuzzy sublattices of L. It is well-known that an arbitrary intersection of fuzzy sublattices is a fuzzy sublattice. A fuzzy sublattice of L generated by a fuzzy set μ is defined as the smallest fuzzy sublattice of L containing μ and is denoted by $[\mu]$. Clearly,

$$[\mu] = \bigcap\{\eta \in \mathcal{L}(L)/\mu \subseteq \eta\}.$$

The symbol $[A]$ denotes the sublattice generated by a subset A of$L$.

Theorem 2.2 [11]**.** Let $\mu \in \mathcal{F}(L)$, Subsequently, μ is a fuzzy sublattice of L iff each level subset μ_t for $t \in \operatorname{Im}\mu$ is a sublattice of L.

Equivalently, a fuzzy set μ in L is a fuzzy sublattice of L iff each nonempty level subset μ_t is a sublattice of L.

The definitions of a fuzzy ideal (dual ideal) and fuzzy prime ideal (dual ideal) in a lattice L are given by Ajmal and Kohli [20].

Definition 2.3 [11]. A fuzzy sublattice μ of a lattice L is called

(i) A fuzzy ideal of L if $x \leq y$ in L implies $\mu(x) \geq \mu(y)$;

(ii) A fuzzy dual ideal of L if $x \leq y$ in L implies $\mu(x) \leq \mu(y)$.

Let $\mathcal{I}(L)$ and $\mathcal{D}(L)$ denote the sets of the fuzzy and fuzzy dual ideals of L, respectively. The fuzzy ideal $(\theta]$ generated by a fuzzy set θ is defined as the smallest fuzzy ideal L that contains θ. Similarly, the fuzzy dual ideal $[\gamma)$ generated by the fuzzy set γ is defined as the smallest fuzzy dual ideal of L containing γ. Because the set of all fuzzy ideals (dual ideals) of L is closed at arbitrary intersections,

$$(\theta] = \bigcap\{\eta \in \mathcal{I}(L)/\theta \subseteq \eta\},$$
$$[\gamma) = \bigcap\{\eta \in \mathcal{D}(L)/\gamma \subseteq \eta\}.$$

The same symbols $(A]$ and $[A)$ denote an ideal and a dual ideal generated by subset A of L, respectively.

Theorem 2.4 [11]. Let $\mu \in \mathcal{L}(L)$. Then, μ is the fuzzy ideal (fuzzy dual ideal) of L if and only if each level subset μ_t for $t \in \operatorname{Im}\mu$ is an ideal (dual ideal) of L.

Equivalently, a fuzzy sublattice μ in L is a fuzzy ideal (dual ideal) of L if and only if each nonempty level subset μ_t is an ideal (dual ideal) of L.

If μ is the fuzzy ideal (fuzzy dual ideal) of L, then the ideal (dual ideal) μ_t is the level ideal (dual ideal) of L.

Theorem 2.5 [11]. (i) A fuzzy ideal μ of L is called a fuzzy prime ideal if

$$\mu(xy) \leq \max\{\mu(x), \mu(y)\} \quad \forall\, x, y \in L;$$

(ii) A fuzzy dual ideal μ of L is called a fuzzy prime dual ideal if

$$\mu(x+y) \leq \max\{\mu(x), \mu(y)\} \quad \forall\, x, y \in L.$$

Let $\mathcal{FP}(L)$ and $\mathcal{FDP}(L)$ denote the sets of all the fuzzy prime ideals and fuzzy prime dual ideals of L, respectively.

Theorem 2.6 [11]. Let $\mu \in \mathcal{I}(L)$ $(\mu \in \mathcal{D}(L))$. μ is a fuzzy prime ideal (fuzzy prime dual ideal) of L if and only if each level ideal (dual ideal) μ_t for $t \in \operatorname{Im}\mu$ is the prime ideal (dual prime ideal) of L.

Equivalently, a fuzzy ideal (dual ideal) μ is a fuzzy prime ideal (fuzzy prime dual ideal) of L if and only if each nonempty-level ideal μ_t is a prime ideal (dual prime ideal) of L.

Wong's introduction of fuzzy points significantly advanced the field of fuzzy topologies, enabling numerous findings on countability, separability, compactness, and convergence using this concept. Mordeson et al. [8] defined the concept of a fuzzy coset in a fuzzy group using fuzzy point.

Definition 2.7. For any $x \in X$, a fuzzy point x_α in set X is a fuzzy set given by

$$x_\alpha(a) = \left\{\begin{matrix} \alpha, & \text{if } a = x, \\ 0, & \text{otherwise,} \end{matrix}\right\}$$

where $0 < \alpha < 1$. The fuzzy point x_α is said to have support x and a value α.

For a distributive lattice L, let P be a set of fuzzy points in L. That is,

$$P = \{x_\alpha/0 < \alpha < 1,\, x \in L\}.$$

Define a relation "$\leq$" in P as: $x_\alpha \leq y_\beta$ if and only if $x \leq y$ in L and $\alpha \leq \beta$ in $(0, 1)$. It can be verified that P is a Poset with respect to this relation.

3. Some Characterizations

In this section, we describe certain characterizations of the fuzzy ideal and fuzzy dual ideal of a lattice. For these characterizations, we use the concept of a strong-level subset of a fuzzy set. Strong-level subsets first appeared in [21], where the modularity of the lattice of the fuzzy normal subgroups of a group was established. The notion of strong-level subsets effectively replaces the notion of level subsets in fuzzy group theory studies. The application of strong-level subsets simplifies the proofs of results considerably and often removes the need for the sup property restriction. Head [22, 23], who establishing his well-known Metatheorem, defined the Rep function using strong level subsets.

Theorem 3.1. We assume that $\mu \in \mathcal{L}(L)$. Subsequently, the following are equivalent:

(i) μ is a fuzzy ideal (dual ideal) of L.

(ii) Each strong level subset $\mu_t^{>}$, for $t < \sup\mu$, is an ideal (dual ideal) of L.

(iii) Each nonempty strong-level subset $\mu_t^>$, is an ideal (dual ideal) of L.

Proof. (i)⇒(ii): Let $t < \sup \mu$ and $x, y \in \mu_t^>$. Subsequently, $\mu(x) > t$ and $\mu(y) > t$. As μ is a fuzzy sublattice,

$$\mu(x+y) \geq \min\{\mu(x), \mu(y)\} > t,$$
$$\mu(xy) \geq \min\{\mu(x), \mu(y)\} > t.$$

Hence $x+y$, $xy \in \mu_t^>$. Thus $\mu_t^>$ is a sublattice of L. Next, let $x \in \mu_t^>$ and $y \leq x$ be L. As μ is the fuzzy ideal of L, $\mu(x) \leq \mu(y)$. We thus get $\mu(y) \geq \mu(x) > t$. Thus, $y \in \mu_t^>$. Consequently, $\mu_t^>$ is the ideal value of L.

(ii)⇒(iii): Let $\mu_t^>$ be a nonempty strong-level subset of μ. Suppose $z \in \mu_t^>$. Subsequently,

$$t < \mu(z) \leq \sup \mu.$$

Therefore, from (ii), $\mu_t^>$ is ideal for L.

(iii)⇒(i): First, we assume that $\exists\, x, y \in L$ such that $\mu(x+y) < \min\{\mu(x), \mu(y)\}$. Subsequently, by setting $\mu(x+y) = t$, we obtain

$$\min\{\mu(x), \mu(y)\} > t.$$

This implies $\mu(x) > t$ and $\mu(y) > t$. Thus, $x, y \in \mu_t^>$. As $\mu_t^>$ is a sublattice of L,

$$x + y \in \mu_i^>.$$

This result contradicts that $\mu(x+y) = t$. We thus have,

$$\mu(x+y) \geq \min\{\mu(x), \mu(y)\} \quad \forall\, x, y \in L.$$

Similarly,

$$\mu(xy) \geq \min\{\mu(x), \mu(y)\} \quad \forall\, x, y \in L.$$

In contrast, suppose $\exists\, x, y \in L$ such that $x \leq y$ and $\mu(x) < \mu(y)$. We set $\mu(x) = t$. Subsequently,

$$y \in \mu_i^> \text{ but } x \notin \mu_t^{\cdot}$$

This is a contradiction, because $\mu_t^>$, being nonempty, is an ideal of L. Therefore, μ is the fuzzy ideal of L. This completes the proof.

The proof for the dual ideal is similar; hence, it is omitted. □

The next theorem comes as a consequence of Theorem 3.4 and Theorem 4.1.

Theorem 3.2. We assume that $\mu \in \mathcal{L}(L)$. Subsequently, the following are equivalent:

(i) μ is a fuzzy ideal (fuzzy dual ideal) of L.

(ii) Each level subset μ_t, for $t \in \operatorname{Im} \mu$, is an ideal (dual ideal) of L.

(iii) Each nonempty level subset μ_t is an ideal (dual ideal) of L.

(iv) Each strong level subset $\mu_t^>$, for $t < \sup \mu$, is an ideal (dual ideal) of L.

(v) Each nonempty strong-level subset $\mu_t^>$ is an ideal (dual ideal) of L.

The following result provides a simple characterization of the fuzzy prime ideal of L. The proof is straightforward; hence, it is omitted here.

Theorem 3.3. Let μ be a fuzzy ideal of L. Subsequently, μ is the fuzzy prime ideal iff

$$\mu(xy) = \mu(x) \text{ or } \mu(y) \quad \forall\, x, y \in L.$$

The next theorem provides the equivalent conditions for μ to become the fuzzy prime ideal (dual ideal) of L. The proof that is similar to Theorem 4.1 is omitted.

Theorem 3.4. Let $\mu \in \mathcal{I}(L)$ $\{\mu \in \mathcal{D}(L)\}$. Subsequently, the following are equivalent:

(i) μ is a fuzzy prime ideal (prime dual ideal) of L.

(ii) Each level ideal μ_t, for $t \in \operatorname{Im} \mu$, is a prime ideal (dual ideal) of L.

(iii) Each nonempty level ideal μ_t is a prime ideal (dual ideal) of L.

(iv) Each strong-level subet $\mu_t^>$, for $t < \sup \mu$, is a prime ideal (dual ideal) of L.

(v) Each nonempty strong-level subset $\mu_t^>$ is a prime ideal (dual ideal) of L.

The above characterizations help establish that the union of an ascending chain of fuzzy ideals in a lattice is a fuzzy ideal. The same applies for the dual fuzzy ideals of L. For this purpose, we provide a more general result regarding the union of a directed family of fuzzy ideals in a lattice.

Theorem 3.5. Let $\{\mu_t\}_{t\in A}$ be a family of fuzzy ideals in lattice L directed under fuzzy-set inclusion. Then, the union $\bigcup_{t\in\Lambda}\mu_t$ is a fuzzy ideal of L.

Proof. In view of Theorem 3.2, it is enough to prove that each nonempty strong-level subset $\left(\bigcup_{i\in\Lambda}\mu_i\right)_t^>$ of the fuzzy set $\bigcup_{r\in\Lambda}\mu_r$ is an ideal of L. We know that,

$$\left(\bigcup_{i\in\Lambda}\mu_i\right)_t^> = \bigcup_{i\in\Lambda}(\mu_i)_t^> \quad \forall\, t\in[0,\ 1].$$

Let $x, y\in\bigcup_{i\in\Lambda}(\mu_i)_t^>$ for some $t\in[0,\ 1]$. Then,

$$x\in(\mu_{i_0})_t^> \text{ and } y\in(\mu_{j_0})_t^> \quad \text{for some } i_0 and j_0\in\Lambda.$$

Because $\{\mu_t\}_{t\in A}$ is a directed family of fuzzy ideals, $\exists\, k\in\Lambda$ such that

$$\mu_{i_0}\subseteq\mu_k \text{ and } \mu_{j_0}\subseteq\mu_k.$$

We have

$$\mu_k(x)\geq\mu_{i_0}(x)>t,\ \ \mu_k(y)\geq\mu_{j_0}(y)>t.$$

As μ_k is a fuzzy sublattice, this implies that

$$\mu_k(x+y)>t \quad\text{and}\quad \mu_k(xy)>t.$$

Thus

$$x+y,\quad xy\in(\mu_k)_t^>\subseteq\bigcup_{i\in\Lambda}(\mu_i)_t^>.$$

Thus, $\bigcup_{i\in\Lambda}(\mu_i)_t^>$. Hence, $\left(\bigcup_{i\in\Lambda}\mu_i\right)_t^>$ is a sublattice of L.

Further, let

$$x\leq y \quad\text{and}\quad y\in\bigcup_{i\in\Lambda}(\mu_i)_t^>.$$

Subsequently, $y\in(\mu_p)_t^>$ for some $p\in\Lambda$. As μ_p is a fuzzy ideal, $(\mu_p)_t^>$ is an ideal of L. Hence,

$$x\in(\mu_p)_t^>\subseteq\bigcup_{i\in\Lambda}(\mu_i)_t^>.$$

Thus, $\bigcup_{i\in\Lambda}(\mu_i)_t^>$ and $\left(\bigcup_{i\in\Lambda}\mu_i\right)_t^>$ is ideal for L. Hence, the union $\bigcup_{r\in\Lambda}\mu_r$ of a directed family of fuzzy ideals is the fuzzy ideal in L. □

The following is an immediate corollary:

Corollary 3.6. The union of an ascending chain of fuzzy ideals in L is a fuzzy ideal.

4. Fuzzy Sublattice (Ideal, Dual Ideal) Generated by a Fuzzy Set

In this section, some interesting techniques for generating fuzzy sublattice, fuzzy ideals, and fuzzy dual ideals are provided using the concepts of level and strong-level subsets.

Theorem 4.1 [6]**.** Let $\mu\in\mathcal{F}(L)$. We define the fuzzy sets μ^* and μ^{**} in L as

$$\mu^*(x)=\sup_{t\in\operatorname{Im}\mu}\{t : x\in[\mu_t]\},$$

and

$$\mu^{**}(x)=\sup_{t<\sup\mu}\{t : x\in[\mu_t^>]\} \quad \forall\, x\in L.$$

Then $\mu^*=\mu^{**}=[\mu]$.

Thus, the fuzzy sublattice generated by μ can either be defined using the classical sublattice generated by the level subsets of μ or by the sublattice in L generated by the strong level subsets of μ. We provide the proof of the next result for similar generating methods for fuzzy ideals in L.

Theorem 4.2. Let $\theta\in\mathcal{F}(L)$. We define the fuzzy sets θ^* and θ^{**} in L as

$$\theta^*(x)=\sup_{t\in\operatorname{Im}\theta}\{t : x\in(\theta_t]\},$$

and

$$\theta^{**}(x)=\sup_{t<\sup\theta}\{t : x\in(\theta_t^>]\} \quad \forall\, x\in L.$$

Subsequently, $\theta^*=\theta^{**}=(\theta]$.

Proof. We first prove that θ^{**} is a fuzzy ideal of L. Let x, $y\in L$ such that

$$\theta^{**}(x+y)<\min\{\theta^{**}(x),\theta^{**}(y)\}.$$

Then, $\theta^{**}(x+y)<\theta^{**}(x)$ and $\theta^{**}(x+y)<\theta^{**}(y)$. Therefore, $\exists\, t_o<\sup\theta$ and $s_o<\sup\theta$ such that $x\in(\theta_{t_o}^>]$, $y\in(\theta_{s_o}^>]$ and

$$\theta^{**}(x+y)<t_o,\quad \theta^{**}(x+y)<s_o.$$

If $t_o=s_o$, then x, $y\in(\theta_{t_o}^>]$. This implies $x+y\in(\theta_{t_o}^>]$ as $(\theta_{t_o}^>]$ is ideal for L generated by $\theta_{t_o}^>$. Therefore,

$$t_o\leq\sup_{t<\sup\theta}\{t : x+y\in(\theta_t^>]\}=\theta^{**}(x+y).$$

This contradicts that $\theta^{**}(x+y)<t_o$.

If $t_o<s_o$, $\theta_{s_0}^>\subseteq\theta_{t_0}^>$. Thus, $x, y\in(\theta_{t_o}^>]$. This implies

$x + y \in (\theta^{>}_{t_o}]$ and, consequently, $t_o \leq \sup\limits_{t<\sup\theta}\{t : x + y \in (\theta^{>}_{t}]\} = \theta^{**}(x + y)$. However, this finding is contradictory.

Thus,

$$\theta^{**}(x + y) \geq \min\{\theta^{**}(x), \theta^{**}(y)\} \quad \forall\, x, y \in L.$$

Let $x \leq y$ be in L. Because $(\theta^{>}_{t}]$ is the fuzzy ideal of L, $y \in (\theta^{>}_{t}]$ implies $x \in (\theta^{>}_{t}]$. That is

$$\sup_{t<\sup\theta}\{t : y \in (\theta^{>}_{t}]\} \leq \sup_{t<\sup\theta}\{t : x \in (\theta^{>}_{t}]\}.$$

Thus,

$$\theta^{**}(y) \leq \theta^{**}(x).$$

Therefore, θ^{**} is the fuzzy ideal of L.

Now, we prove that $\theta \subseteq \theta^{**}$. Suppose that $\theta(x) > \theta^{**}(x)$ for $x \in L$. Choose a real number $t_o \in [0, 1]$, such that

$$\theta(x) > t_0 > \theta^{**}(x).$$

Subsequently, $x \in \theta^{>}_{t_o} \subseteq (\theta^{>}_{t_o}]$. This implies

$$t_o \leq \sup_{t<\sup\theta}\{t : x \in (\theta^{>}_{t}]\} = \theta^{**}(x),$$

This contradicts $\theta^{**}(x) < t_o$. Thus,

$$\theta \subseteq \theta^{**}.$$

Next, we establish that θ^{**} is the least fuzzy ideal of L containing θ. Suppose that η is a fuzzy ideal of L such that $\theta \subseteq \eta$. Suppose that $\exists\, x \in L$ such that

$$\theta^{**}(x) > \eta(x).$$

Let s be a real number, such that

$$\theta^{**}(x) > s > \eta(x).$$

Thus, $\eta(x) < s$ implies that $x \notin \eta^{>}_{s}$. Moreover, $s < \theta^{**}(x)$ implies that $\exists\, t_o < \sup\theta$ such that $x \in (\theta^{>}_{t_o}]$ and $s < t_o$. Since $\theta \subseteq \eta$ and η is a fuzzy ideal of L, therefore by Theorem 4.2 we get

$$x \in (\theta^{>}_{t_o}] \subseteq (\eta^{>}_{t_o}] = \eta^{>}_{t_o}.$$

Further, $s < t_o$ implies $x \in \eta^{>}_{t_o} \subseteq \eta^{>}_{s}$. This finding contradicts $x \notin \eta^{>}_{s}$. Hence,

$$\theta^{**}(x) \leq \eta(x) \quad \forall\, x \in L.$$

That is, θ^{**} is the least fuzzy ideal of L containing θ. Thus,

$$\theta^{**} = (\theta].$$

We now establish $\theta^{*} = \theta^{**}$.

Let $x \in L$. To prove that $\theta^{**}(x) \leq \theta^{*}(x)$, let $t < \sup\theta$ such that $x \in (\theta^{>}_{t}]$. We claim that $\exists\, t_o \in \operatorname{Im}\theta$ such that $t \leq t_o$ and $x \in (\theta_{t_o}]$. Consider two cases:

If $t \in \operatorname{Im}\theta$, then:

$$x \in (\theta^{>}_{t}] \subseteq (\theta_t].$$

If $t \notin \operatorname{Im}\theta$, then: Then, because $t < \sup\theta$, $\exists\, s \in \operatorname{Im}\theta$ such that

$$t < s \leq \sup\theta \quad \text{and} \quad \theta^{>}_{t} = \theta_s.$$

Thus $x \in (\theta^{>}_{t}] = (\theta_s]$.

Consequently, for each $t \in \{t < \sup\theta / x \in (\theta^{>}_{t}]\}$, $\exists\, t_o \in \{t \in \operatorname{Im}\theta / x \in (\theta_t]\}$ such that $t \leq t_o$. Therefore,

$$\sup_{t<\sup\theta}\{t / x \in (\theta^{>}_{t}]\} \leq \sup_{t\in\operatorname{Im}\theta}\{t / x \in (\theta_t]\}.$$

That is,

$$\theta^{**}(x) \leq \theta^{*}(x) \quad \forall\, x \in L.$$

Suppose $\theta^{**}(x) < \theta^{*}(x)$ for $x \in L$. Then by the definition of $\theta^{*}(x)$, $\exists\, t_o \in \operatorname{Im}\theta$, $x \in (\theta_{t_o}]$ such that

$$\theta^{**}(x) < t_o.$$

Because $\theta^{**}(x)$ is the intersection of all the fuzzy ideals of L containing θ, $\exists$ a fuzzy ideal η of L such that $\theta \subseteq \eta$ and $\eta(x) < t_o$. Currently, $\theta \subseteq \eta$ implies that:

$$\theta_{t_o} \subseteq \eta_{t_o}.$$

Therefore,

$$x \in (\theta_{t_o}] \subseteq \eta_{t_o}.$$

where η denotes the fuzzy ideal of L. This implies

$$\eta(x) \geq t_o.$$

However, this finding is contradictory. Hence,

$$\theta^{**}(x) \geq \theta^{*}(x) \quad \forall\, x \in L.$$

Thus,

$$\theta^{**} = \theta^{*}.$$

This completes the proof. □

The following results provide similar techniques for generating a fuzzy dual ideal using a fuzzy set in L.

Theorem 4.3. Let $\gamma \in \mathcal{F}(L)$, we define the fuzzy sets γ^* and γ^{**} in L as

$$\gamma^*(x) = \sup_{t \in \operatorname{Im}\gamma} \{t : x \in [\gamma_t)\},$$

and

$$\gamma^{**}(x) = \sup_{t < \sup \gamma} \{t : x \in [\gamma_t^>)\} \quad \forall\, x \in L.$$

Then $\gamma^* = \gamma^{**} = [\gamma)$.

5. Fuzzy Prime Ideal Theorem

The notion of fuzzy point and quasi-coincident (overlapping) fuzzy sets is crucial in the studies of fuzzy topological spaces. The idea of disquasi-coincident fuzzy sets emerged from the set theory that two subsets of a set are disjoint (non-intersecting) iff one is contained in the complement of the other. However, in fuzzy set theory, the implication is not in either way. In other words, a fuzzy set contained in the complement of another fuzzy set may or may not be disjointed from it. This implies the notion of disquasi-coincident is more general than that of disjoint fuzzy sets. Pu and Liu [19] replaced the notion of disjointness with disquasi-coincident in their studies and thus developed the theory of fuzzy topology.

In fuzzy group theory and all other branches of fuzzy algebraic structures, the concept of a fuzzy point is rarely utilized, with a few exceptions. This contrasts with classical group theory, in which a point and its related notions of belonging are an integral part of the subject's development. In the fuzzy group theory, one of the few places where the notion of a fuzzy point is applied is forming fuzzy cosets [8] and the other place is in [24], where a pointwise characterization of the normality of an L-subgroup in an L-group is provided. A fuzzy point is a fuzzy set that assumes a non-zero value only at a single point, which is called the support of that fuzzy point. Here, we use the concepts of a fuzzy point and overlapping fuzzy sets to establish an important result. For a fuzzy set μ in a set X properly containing another fuzzy set η, there exist infinitely many fuzzy points x_a that do not overlap with η and overlap with μ. That is $x_\alpha \notin \eta$ and $x_\alpha \, q \, \mu$.

In this section, we establish the fuzzy prime ideal theorem using Zorn's lemma. First, we define overlapping and non-overlapping fuzzy sets. Here, the fuzzy set 1_X in set X is defined as a constant fuzzy set with all truth values 1 in L.

Definition 5.1 [19]**.** Let μ and η be fuzzy sets in a set X. μ and η are said to be quasi-coincident (or overlapping) if there exists x in X such that

$$\mu(x) + \eta(x) > 1.$$

This is denoted by

$$\mu \, q \, \eta.$$

In contrast, μ and η are said to be disquasi-coincident (non-overlapping) fuzzy sets if, for all x in X,

$$\mu(x) + \eta(x) \leq 1.$$

This is expressed as $\mu + \eta \leq 1_X$ or $\mu \notin \eta$.

If μ is replaced by a fuzzy point x_α, then x_α is quasi-coincident with η if for some x in X, $\alpha + \eta(x) > 1$. This is expressed as $x_\alpha \, q \, \eta$,

In the prime ideal theorem of lattice theory, the ideal and dual ideals of the lattice are considered to be disjoint. However, to obtain the fuzzy version of this theorem, in the hypothesis, we replace the concept of the disjoint fuzzy ideal and fuzzy dual ideal by the disquasi-coincident (non-overlapping) fuzzy ideal and fuzzy dual ideal. To fix the notation in the following theorem, the symbol $\langle \theta_{t_o}, x \rangle$ is the ideal L generated by $\theta_{t_o} \cup \{x\}$.

Theorem 5.2. Let L be a distributive lattice, μ be a fuzzy ideal in L and η be a fuzzy dual ideal in L such that $\mu + \eta \leq 1_L (\mu \notin \eta)$. Subsequently, there exists a fuzzy prime ideal θ in L such that $\mu \subseteq \theta$ and $\theta + \eta \leq 1_L (\theta \notin \eta)$.

Proof. Let $FI(\mu, \eta)$ be the family (Poset) of all fuzzy ideals of L containing μ and non-overlapping with η. That is,

$$FI(\mu, \eta) = \{\alpha : \alpha \in \mathcal{I}(L),\ \mu \subseteq \alpha \ \text{ and } \ \alpha \notin \eta\}.$$

Let $\Omega = \{\alpha_i\}$ be a chain in $FI(\mu, \eta)$ and

$$\alpha = \bigcup \Omega = \bigcup_i \alpha_i.$$

According to Theorem 3.6, α is the fuzzy ideal of L. Clearly, $\mu \subseteq \alpha$, α and η are nonoverlapping $(\alpha \notin \eta)$. Therefore,

$$\alpha = \cup \Omega \in FI(\mu, \eta).$$

$FI(\mu,\eta)$ is a poset in which each chain has an upper bound. From Zorn's lemma, $FI(\mu,\eta)$ has a maximal element θ. Subsequently, θ is the fuzzy ideal of L, $\mu \subseteq \theta$ and $\theta \not q\, \eta$. We prove that θ is the fuzzy prime ideal of L. By contrast, suppose θ is not a fuzzy prime ideal of L. Subsequently, from Theorem 4.3, $\exists\, x, y \in L$ such that

$$\theta(xy) > \theta(x) \quad \text{and} \quad \theta(xy) > \theta(y).$$

If $\theta(xy) = t_o$, then $x \notin \theta_{t_o}$, $y \notin \theta_{t_o}$ and $xy \in \theta_{t_o}$. Define fuzzy set θ_x^* as

$$\theta_x^* : L \to [0,\ 1],$$
$$\theta_x^*(a) = \begin{cases} \theta(a), & \text{if } a \in \theta_{t_o}, \\ t_o, & \text{if } a \in \langle \theta_{t_o},\ x\rangle \sim \theta_{t_o}, \\ \theta(a), & \text{if } a \in L \sim \langle \theta_{t_o},\ x\rangle. \end{cases}$$

Similarly, we define the fuzzy set θ_y^* as

$$\theta_y^* : L \to [0,\ 1],$$
$$\theta_y^*(a) = \begin{cases} \theta(a), & \text{if } a \in \theta_{t_o}, \\ t_o, & \text{if } a \in \langle \theta_{t_a},\ y\rangle \sim \theta_{t_o}, \\ \theta(a), & \text{if } a \in L \sim \langle \theta_{t_o},\ y\rangle. \end{cases}$$

Since each level subset of θ_x^* and θ_y^* is an ideal of L, therefore by Theorem 3.2, θ_x^*, θ_y^* are fuzzy ideals of L. Moreover,

$$\theta(x) < t_o = \theta_x^*(x) \quad \text{and} \quad \theta(y) < t_o = \theta_y^*(y).$$

Therefore,

$$\mu \subseteq \theta \subset \theta_x^* \quad \text{and} \quad \mu \subseteq \theta \subset \theta_y^*.$$

Currently, θ is maximal in $FI(\mu,\eta)$, we obtain $\theta_x^*, \theta_y^* \notin FI(\mu,\eta)$. Therefore, the fuzzy sets θ_x^* and η overlap and the fuzzy sets θ_y^* and η overlap. That is,

$$\theta_x^*\, q\, \eta \ \text{ and } \ \theta_y^*\, q\, \eta$$

Thus, $\exists\, u_1, u_2 \in L$ such that

$$\theta_x^*(u_1) + \eta(u_1) > 1 \ \text{ and } \ \theta_y^*(u_2) + \eta(u_2) > 1.$$

Because $\theta \not q\, \eta$, we have

$$\theta_x^*(u_1) = t_0 \ \text{ and } \ \theta_y^*(u_2) = t_0.$$

That is,

$$u_1 \in \langle \theta_{t_o}, x\rangle \sim \theta_{t_o},$$

and

$$u_2 \in \langle \theta_{t_a}, y\rangle \sim \theta_{t_o}.$$

We have $t_o + \eta(u_1) > 1$ and $t_o + \eta(u_2) > 1$. That is,

$$\eta(u_1) > 1 - t_o = s_o, \ \ \eta(u_2) > 1 - t_o = s_o.$$

This implies $u_1, u_2 \in \eta_{s_o}^{>}$. As η is the fuzzy dual ideal of L,

$$\eta(u_1 u_2) > s_o.$$

Because $u_1 \in \langle \theta_{t_o}, x\rangle \sim \theta_{t_o}$ and $u_2 \in \langle \theta_{t_o}, y\rangle \sim \theta_{t_o}$,

$$u_1 \le v_1 + x \ \text{ and } \ u_2 \le v_2 + y \quad \text{ for some } \ v_1 and v_2 \in \theta_{t_o}.$$

Thus,

$$u_1 u_2 \le v_1 v_2 + v_1 y + v_2 x + xy \in \theta_{t_o}$$

because θ_{t_o} is ideal for L and $xy \in \theta_{t_o}$. This gives $u_1 u_2 \in \theta_{t_o}$ and therefore, $\theta(u_1 u_2) \ge t_o$. Finally,

$$\theta(u_1 u_2) + \eta(u_1 u_2) > t_o + s_o = 1.$$

Therefore, θ and η are fuzzy sets that overlap with $u_1 u_2$. That is,

$$\theta\, q\, \eta.$$

This result contradicts that $\theta \not q\, \eta$. Thus, we conclude that θ is the fuzzy prime ideal of L. □

A fuzzy set μ in L is considered appropriate if $\mu \ne 1_L$; that is, $\exists\, x \in L$ such that $\mu(x) < 1$. The next result is an interesting application of fuzzy prime ideal theorem.

Theorem 5.3. Every proper fuzzy ideal in a distributive lattice L is the intersection of fuzzy prime ideals of L.

Proof. Let μ be a proper fuzzy ideal in distributive lattice L. Let

$$T = \{\eta/\eta \in \mathcal{FP}(L)\ \text{, such that } \mu \subseteq \eta\}.$$

Then clearly,

$$\mu \subseteq \bigcap_{n \in T} \eta.$$

We assume that $\mu \ne \bigcap_{\eta \in T} \eta$. Then, $\exists\, x \in L$ such that

$$\mu(x) < \Big(\bigcap_{\eta \in T} \eta\Big)(x).$$

Currently, we choose a fuzzy point x_α that is non-overlapping (disquasi-coincident) with μ and overlapping (quasi-coincident) with $\bigcap_{\eta\in T}\eta$. That is, we choose α such that

$$x_\alpha \not{q}\, \mu \text{ and } x_\alpha q\left(\bigcap_{\eta\in T}\eta\right). \qquad (*)$$

This statement also implies the following:

$$\alpha+\mu(x)\le 1 \text{ and } \alpha+\left(\bigcap_{\eta\in T}\eta\right)(x)>1.$$

This is possible if we choose $\alpha\in[0,1]$ such that

$$\mu(x)\le 1-\alpha<\left(\bigcap_{\eta\in T}\eta\right)(x).$$

We now consider the dual ideal $[x)$ of L generated by x. Construct a fuzzy set γ in L given by

$$\gamma(z)=\begin{cases}\alpha, & \text{if } z\in[x),\\ 0, & \text{if } z\in L\sim[x).\end{cases}$$

γ is clearly a fuzzy dual ideal of L with level subsets $[x)$ and L (Theorem 2.4). To use the fuzzy prime ideal theorem, we must demonstrate that γ does not overlap with the fuzzy ideal μ. That is,

$$\mu \not{q}\, \gamma.$$

Consider $z\in L$. Here, we consider two cases.

Case 1. $z\in[x)$. Then, $x\le z$ and $\gamma(z)=\alpha$. As μ is the fuzzy ideal of L,

$$\mu(z)\le\mu(x).$$

Thus, $(\mu+\gamma)(z)=\mu(z)+\gamma(z)\le\mu(x)+\alpha\le 1$.

Case 2. $z\notin[x)$. Then

$$(\mu+\gamma)(z)=\mu(z)+\gamma(z)=\mu(z)+0\le 1.$$

Therefore, $(\mu+\gamma)(z)\le 1\ \forall\, z\in L$. Hence

$$\mu+\gamma\le 1_L.$$

Therefore, according to the fuzzy prime ideal theorem, there exists a fuzzy prime ideal θ of L such that $\mu\subseteq\theta$ and $\theta \not{q}\,\gamma$. That is, θ and γ are non-overlapping. Clearly, $\theta\in T$, which implies that

$$\bigcap_{\eta\in T}\eta\subseteq\theta. \qquad (**)$$

Now,

$$\begin{aligned}(\theta+\gamma)(x)&=\theta(x)+\gamma(x)\\ &=\theta(x)+\alpha && (\text{as } x\in[x))\\ &\ge\left(\bigcap_{\eta\in T}\eta\right)(x)+\alpha && (\text{by } (**))\\ &>1. && (\text{by } (*))\end{aligned}$$

Therefore, θ overlaps with γ at x, contradicting the fact that θ and γ are non-overlapping ($\theta \not{q}\,\gamma$). Hence

$$\mu=\bigcap_{n\in T}\eta.$$

□

Another application of the fuzzy prime ideal theorem is presented in the next result, where the existence of a fuzzy prime ideal in a sublattice of L ensures the existence of a corresponding fuzzy prime ideal in the lattice L.

Theorem 5.4. Let L be a distributive lattice and L' be a sublattice of L. If μ' is a fuzzy set in L such that $(\mu')_{L'}$ (μ' restricted to L') is the fuzzy prime ideal of L', then there exists a fuzzy prime ideal θ in L such that $(\theta)_{L'}=(\mu')_{L'}$.

Proof. Let $\mu=(\mu']_L$ be the fuzzy ideal in L generated by μ'. Then, by Theorem 4.2,

$$\mu(x)=\sup_{t\in\operatorname{Im}\mu'}\{t/x\in(\mu'_t]\},$$

where $(\mu'_t]$ is the ideal in L generated by μ'_t.
Let $\eta=[1_L\sim\mu')_L$ be the fuzzy dual ideal in L generated by fuzzy set $1_L\sim\mu'$. Subsequently, by Theorem 4.3,

$$\eta=\sup_{t\in\operatorname{Im}(1_L\sim\mu')}\{t/x\in[(1_L\sim\mu')_t)\},$$

where $[(1_L\sim\mu')_t)$ is the dual ideal in L generated by fuzzy set $(1_L\sim\mu')_t$.
We first claim that μ and η are non-overlapping fuzzy sets. That is, $\mu \not{q}\,\eta$. Suppose on the contrary, $\mu\, q\, \eta$. That is, $\exists\, x\in L$ such that

$$\mu(x)+\eta(x)>1.$$

Subsequently, by defining μ, $\exists\, t_1\in\operatorname{Im}\mu'$ such that $x\in(\mu'_{t_1}]$ and

$$t_1>1-\eta(x).$$

Similarly, from the definition of η, $\exists\, t_2\in\operatorname{Im}(1_L\sim\mu')$ such that $x \in [(1_L \sim \mu')_{t_2})$ and $t_2 > 1 - t_1$. Thus,

$$t_1 + t_2 > 1.$$

According to the definitions of ideal and dual ideals, $\exists\, u \in (1_L \sim \mu')_{t_2}$ and $v \in \mu'_{t_1}$ such that

$$u \leq x \leq v.$$

Here, $u \leq v$ and $v \in \mu'_{t_1}$ are ideal for L'. Thus, $u \in \mu'_{t_1}$, Thus,

$$\mu'(u) \geq t_1 > 1 - t_2.$$

This implies

$$1 \sim \mu'(u) < t_2.$$

This contradicts $u \in (1_L \sim \mu')_{t_2}$. Thus, $\forall\, x \in L$,

$$\mu(x) + \eta(x) \leq 1.$$

That is, $\mu \not\between \eta$.

Therefore, according to the fuzzy prime ideal theorem, there exists a fuzzy prime ideal θ in L such that $\mu \subseteq \theta$, θ and η are non-overlapping (that is, $\theta \not\between \eta$). We shall prove that

$$(\theta)_{L'} = (\mu')_{L'}.$$

Suppose there exists $x \in L'$ such that

$$\mu'(x) > \theta(x) \geq \mu(x).$$

Subsequently, $\exists$ as a fuzzy ideal α of L such that $\alpha(x) < \mu'(x)$. This contradicts the fact that $\mu' \subseteq \alpha$. Thus,

$$\mu'(x) \leq \theta(x) \quad \forall\, x \in L'.$$

Suppose $\mu'(x) < \theta(x)$ for some $x \in L' \subseteq L$. Consider

$$\begin{aligned}(\theta + \eta)(x) &= \theta(x) + \eta(x)\\ &> \mu'(x) + \eta(x)\\ &= \mu'(x) + \inf\{\beta(x)/\beta \in \mathcal{D}(L), 1_L \sim \mu' \subseteq \beta\}\\ &\geq \mu'(x) + (1 \sim \mu'(x))\\ &= 1.\end{aligned}$$

This result contradicts that $\theta \not\between \eta$. Thus,

$$(\theta)_{L'} = (\mu')_{L'}.$$

Hence the proof. □

In the following theorem, we establish the important fact that the classical prime ideal theorem easily follows the fuzzy prime ideal theorem.

Theorem 5.5. Let L be a distributive lattice, I an ideal in L and F a dual ideal in L such that $I \cap F = \emptyset$. Subsequently, $\exists$ as a prime ideal J in L such that $I \subseteq J$ and $J \cap F = \emptyset$.

Proof. Let I be an ideal and F be a dual ideal in L such that $I \cap F = \emptyset$. Subsequently, by Theorem 3.1, χ_I (the characteristic function of I) is a fuzzy ideal, and χ_F is a fuzzy dual ideal of L. Suppose $\exists\, x \in L$ such that

$$(\chi_I + \chi_F)(x) > 1.$$

Subsequently, from the definitions of χ_I and χ_F, we have $\chi_I(x) = \chi_F(x) = 1$. That is, $x \in I \cap F$. This contradicts because $I \cap F = \emptyset$. Therefore, $\chi_I + \chi_F \leq 1_L$. Therefore, using the fuzzy prime ideal theorem (Theorem 5.1), $\exists$ as a fuzzy prime ideal θ of L such that $\chi_I \subseteq \theta$ and $\theta + \chi_F \leq 1_L$. Let

$$J = Supp\ \theta = \{x \in L/\theta(x) > 0\} = \theta_o^{>}.$$

Since θ is a fuzzy prime ideal of L, therefore, by Theorem 3.4, $\theta_o^{>}$, (i.e., $Supp\ \theta$) is the primary ideal of L. As $\chi_I \subseteq \theta$, we have $I \subseteq Supp\ \theta = J$. Moreover, since $\theta + \chi_F \leq 1_L$, we have $Supp\ \theta \cap F = \emptyset$. In other words, $J \cap F = \emptyset$. This completes the proof of the prime-ideal theorem. □

6. Conclusion

The prime ideal theorem is crucial in the distributive lattice theory. In this study, we provide a fuzzy prime ideal theorem along with two of its applications, which are extensions of the results from classical lattice theory.

Here, we obtain a fuzzy version of the prime ideal theorem using Zorn's lemma. Therefore, the axiom of choice implies a fuzzy prime ideal theorem. As proved in Theorem 5.5, classical prime ideal theorem follows from fuzzy prime ideal theorem. Thus, the fuzzy prime ideal theorem lies strictly between the axiom of choice and prime ideal theorem of the classical lattice. If it is confirmed that the fuzzy version of the prime ideal theorem is equivalent to the axiom of choice, this would mark a significant milestone in the field.

The author also suggests the following for future work. First, to develop the theory of fuzzy lattices, we suggest that the evaluation lattice, the interval $[0, 1]$, be replaced with a more general lattice L, that is, ordinary fuzzy sets should be replaced by lattice-valued fuzzy sets (L-fuzzy sets), as introduced by Goguen [9]. This will extend the work in this field beyond the preview of the metatheorem introduced by Head [22, 23]. Second, researchers are working on fuzzy algebraic structures that replace the parent algebraic structure with the fuzzy algebraic structure. This facilitates a more seamless extension of concepts such as the maximal ideal in a lattice.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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