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Transitive Closure of Vague Soft Set Relations and its Operators

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Abstract

A vague soft set is a mapping from a parameter set to the collection of vague subsets of the universal set. In this study, a vague soft relation is presented based on the Cartesian product of vague soft sets. The basic properties of these relations are studied to explain the concept of transitive closure of a vague soft relation. The symmetric, reflexive, and transitive closures of a vague soft set are introduced followed by examples to illustrate these relations. The concepts are further extended by proposing some of their properties. The existence and uniqueness of the transitive closure of a vague soft relation are established, and an algorithm to compute the transitive closure of a vague soft relation is also provided.

Keywords: Vague soft set, Transitive closure, Symmetric closure, Fuzzy set

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1. Introduction

Many researchers in economics, engineering, environmental sciences, social sciences, medical sciences, business, management, and numerous other fields encounter the modeling complexities presented by uncertain data on a daily basis. However, classical mathematical methods are not always effective because the uncertainties in these domains may be of various types. Probability theory, fuzzy set theory [1–4], intuitionistic fuzzy set theory [5], multi-fuzzy set theory [6,7], vague set theory [8,9], and interval mathematics [10,11] are often useful mathematical tools for describing uncertainty. Molodtsov [12] introduced the concept of a soft set for uncertain data. Maji and his colleagues [13,14] used the soft set theory in decision-making problems and introduced the concept of a fuzzy soft set [15]. Soft sets have been studied by many researchers, such as fuzzy soft sets [16–18], intuitionistic fuzzy soft sets [19–22], vague soft sets [23–28], multi-fuzzy soft sets [29–35], and vague soft set relations and functions [36]. Babitha and Sunil [37] introduced the concept of soft set relations and functions. Agarwal et al. [38] discussed the concept of relations in generalized intuitionistic fuzzy soft sets. Ibrahim et al. [39] introduced the concept of soft set composition relations and the construction of transitive closure. Park et al. [40] studied some properties of the equivalence of soft set relations, while Su et al. [41] introduced the concept of intuitionistic fuzzy decision-making with similarity measures and the ordered weighted averaging (OWA) operator. In addition, Saxena and Tayal [42] proposed the concept of normalization for the type-2 fuzzy relational data model based on fuzzy functional dependency, using fuzzy functions.

In the real world, vaguely specified data values exist in many applications, such as in data with fuzzy, imprecise, and uncertain properties. Fuzzy set (FS) theory was proposed to handle such vagueness by generalizing the notion of membership in a set. In a FS, each element is assigned a single value in the interval [0,1] reflecting its membership grade. This single value does not allow the separation of for membership evidence and against membership evidence. A vague set is a further generalization of FS. Instead of using point-based membership as in FSs, interval-based membership is used in a vague set. The interval-based membership is more expressive in capturing the vagueness of the data.

The remainder of this paper is organized as follows. In Section 2, basic notions about transitive closure of soft sets are reviewed. In Section 3, the transitive closure of vague soft sets is introduced; some theorems are proved; and examples are provided. In Section 4, certain properties of closure are studied on a vague soft set. The last section summarizes the contributions and highlights future research work.

2. Preliminaries

In this section, some basic concepts of vague, soft, and vague soft sets are briefly reviewed.

2.1 Vague Sets

A vague set [8] over U is characterized by a truth-membership function t_ν and false membership function f_ν,

$$t_\nu : U \to [0,1] \text{ and } f_\nu : U \to [0,1],$$

where for any $u_i \in U$, $t_\nu(u_i)$ is a lower bound on the membership grade of u_i derived from the evidence for u_i; $f_\nu(u_i)$ is a lower bound on the negation of u_i derived from the evidence against u_i; and $t_\nu(u_i) + f_\nu(u_i) \leq 1$. The membership grade of u_i in the vague set is bounded to a subinterval $[t_\nu(u_i), 1 - f_\nu(u_i)]$ of [0, 1]. The vague values $[t_\nu(u_i), 1 - f_\nu(u_i)]$ indicate that the exact membership grade $\mu_\nu(u_i)$ of u_i may be unknown, but it is bounded by $t_\nu(u_i) \leq \mu_\nu(u_i) \leq 1 - f_\nu(u_i)$, where $t_\nu(u_i) + f_\nu(u_i) \leq 1$.

A review of the basic operations of the complement, intersection, and union of a vague set, as defined by Gau and Buehrer [8], are presented next.

Definition 1 [8]**.** The complement of a vague set A is denoted by A^c and is defined by

$$t_{A^c} = f_A,\ 1 - f_{A^c} = 1 - t_A.$$

Definition 2 [8]**.** The intersection of two vague sets A and B is a vague set C, denoted as $C = A \cap B$, with truth-membership and false membership functions related to those of A and B by

$$t_C = \min(t_A, t_B),$$
$$1 - f_C = \min(1 - f_A, 1 - f_B) = 1 - \max(f_A, f_B).$$

Definition 3 [8]**.** The union of two vague sets A and B is a vague set C, denoted as $C = A \cup B$, with truth-membership and false membership functions related to those of A and B by

$$t_C = \max(t_A, t(B),$$
$$1 - f_C = \max(1 - f_A, 1 - f_B) = 1 - \min(f_A, f_B).$$

2.2 Soft Sets

The soft set theory was proposed by Molodtsov [12] to provide an appropriate framework for uncertainty modeling. Molodtsov's definitions of soft sets, soft subsets, complement, and the union of soft sets are presented below. Let U be the universe of discourse, and let E be the universe of all possible parameters related to the objects in U.

Definition 4 [12]**.** Let U be a universal set and let E be a set of parameters. Let $P(U)$ denote the power set of U and $A \subseteq E$. A pair (F, A) is called a soft set over U, where F is the mapping

$$F : A \to P(U).$$

Thus, a soft set over U is a parameterized family of subsets of universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered a set of ε-approximate elements of the soft set (F, A).

Definition 5 [12]**.** Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 6 [12]**.** For two soft sets (F, A) and (G, B) over U, (F, A) is called a soft subset of (G, B) if

(i) $A \subseteq B$,

(ii) $\forall \varepsilon \in A, F(\varepsilon) \subseteq G(\varepsilon)$.

This relationship is denoted as $(F,\ A) \tilde{\subseteq} (G, B)$. In this case, (G, B) is called the soft superset of (F, A).

Definition 7. The complement of a soft set (F, A) is denoted by $(F, A)^c$ and defined by $(F, A)^c = (F^c, \rceil A)$, where F^c : $\rceil A \to P(U)$ is a mapping given by

$$F^c(\alpha) = U - F(\rceil\alpha),\ \forall\alpha \in \rceil A.$$

Definition 8 [12]**.** The union of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C) where $C = A \cup B$, and $\forall\varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} F(\varepsilon), & \text{if} \quad \varepsilon \in A - B, \\ G(\varepsilon), & \text{if} \quad \varepsilon \in B - A, \\ F(\varepsilon) \cup G(\varepsilon), & \text{if} \quad \varepsilon \in A \cap B. \end{cases}$$

Ali et al. [43] proposed a definition of the extended intersection of soft sets as follows:

Definition 9 [43]**.** The extended intersection of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C) where $C = A \cup B$, and $\forall\varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} F(\varepsilon), & \text{if} \quad \varepsilon \in A - B, \\ G(\varepsilon), & \text{if} \quad \varepsilon \in B - A, \\ F(\varepsilon) \cap G(\varepsilon), & \text{if} \quad \varepsilon \in A \cap B. \end{cases}$$

2.3 Vague Soft Sets

By combining a vague set and a soft set, Xu et al. [23,24] proposed a new concept called a vague soft set along with its operations of union and intersection, as specified in the following definitions. Let U be a universe, E be a set of parameters, $V(U)$ be the power set of vague sets on U, and $A \subset eqE$.

Definition 10 [23]**.** A pair $(\tilde{F}, A)$ is called a vague soft set over U, where $\tilde{F}$ is a mapping given by

$$\tilde{F} : A \to V(U).$$

In other words, a vague soft set over U is a parameterized family of vague sets of universe U. For $\varepsilon \in A$, $\mu_{\tilde{F}(\varepsilon)} : U \to [0, 1]^2$ is regarded as the set of ε approximate elements of the vague soft set $(\tilde{F}, A)$.

Definition 11 [23]**.** The union of two vague soft sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ over a universe U is a vague soft set denoted by $(\tilde{H}, C)$, where $C = A \cup B$, and

$$t_{\tilde{H}(e)}(x) = \begin{cases} t_{\tilde{F}(e)}(x), & \text{if } e \in A - B, \\ t_{\tilde{G}(e)}(x), & \text{if } e \in B - A, \\ \max\{(t_{\tilde{F}(e)}(x), t_{\tilde{G}(e)}(x)\}, & \text{if } e \in A \cap B, \end{cases}$$

and

$$1 - f_{\tilde{H}(e)}(x) = \begin{cases} 1 - f_{\tilde{F}(e)}(x), & \text{if } e \in A - B, \\ 1 - f_{\tilde{G}(e)}(x), & \text{if } e \in B - A, \\ 1 - \min\left\{f_{\tilde{F}(e)}(x), f_{\tilde{G}(e)}(x)\right\}, & \text{if } e \in A \cap B,. \end{cases}$$

for all $e \in C$ and $x \in U$. This is denoted as $(\tilde{F}, A)\tilde{\cup}(\tilde{G}, B) = (\tilde{H}, C)$.

Definition 12 [23]**.** The intersection of two vague soft sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ over a universe U is a vague soft set denoted by $(\tilde{H}, C)$ where $C = A \cup B$, and

$$t_{\tilde{H}(e)}(x) = \begin{cases} t_{\tilde{F}(e)}(x), & \text{if} e \in A - B, \\ t_{\tilde{G}(e)}(x), & \text{if } e \in B - A, \\ \min\{(t_{\tilde{F}(e)}(x), t_{\tilde{G}(e)}(x)\}, & \text{if } e \in A \cap B, \end{cases}$$

and

$$1 - f_{\tilde{H}(e)}(x) = \begin{cases} 1 - f_{\tilde{F}(e)}(x), & \text{if } e \in A - B, \\ 1 - f_{\tilde{G}(e)}(x), & \text{if } e \in B - A, \\ 1 - \max\{f_{\tilde{F}(e)}(x), f_{\tilde{G}(e)}(x)\}, & \text{if } e \in A \cap B,. \end{cases}$$

for all $e \in C$ and $x \in U$ values. This is denoted by $(\tilde{F},A)\tilde{\cap}(\tilde{G}, B) = (\tilde{H}, C)$.

Definition 13 [24]**.** Let $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be two vague soft sets over U. The restricted union of $(\tilde{F}, A)$ and $(\tilde{G}, B)$ is defined as: the vague soft set $\langle\tilde{H}, C\rangle$, where $C = A \cup B$ and

$$t_{\tilde{H}(e)}(x) = \max\left\{(t_{\tilde{F}(e)}(x), t_{\tilde{G}(e)}(x)\right\},$$
$$1 - f_{\tilde{H}(e)}(x) = 1 - \min\left\{f_{\tilde{F}(e)}(x), f_{\tilde{G}(e)}(x)\right\},$$

for all $e \in C$, and $x \in U$ if $\in A \cap B \neq \phi$; otherwise, $(\tilde{H}, C) \neq \phi_\phi$. This is denoted by $(\tilde{H}, C) = (\tilde{F}, A) \Cup (\tilde{G}, B)$.

Definition 14 [24]**.** Let $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be two vague soft sets over U. The restricted intersection of $(\tilde{F}, A)$ and $(\tilde{G}, B)$ is defined as the vague soft set $\langle\tilde{H}, C\rangle$, where $C = A \cup B$ and

$$t_{\tilde{H}(e)}(x) = \min\left\{(t_{\tilde{F}(e)}(x), t_{\tilde{G}(e)}(x)\right\},$$
$$1 - f_{\tilde{H}(e)}(x) = 1 - \max\left\{f_{\tilde{F}(e)}(x), f_{\tilde{G}(e)}(x)\right\},$$

for all $e \in C$, and $x \in U$ if $\in A \cap B \neq \phi$; otherwise, $(\tilde{H}, C) \neq$

ϕ_ϕ. This is denoted by $(\tilde{H}, C) = (\tilde{F}, A) \Cap (\tilde{G}, B)$.

Definition 15 [36]**.** If a pair $(\tilde{F}, A)$ and $(\tilde{G}, B)$ are two vague soft sets over U, then the Cartesian product of $(\tilde{F}, A)$ and $(\tilde{G}, B)$ is defined as, $(\tilde{F}, A) \times (\tilde{G}, B) = (\tilde{H}, A \times B)$, where $\tilde{H} : A \times B \to V(U \times U)$ and $\tilde{H}(a, b) = F(a) \times G(b)$, where $(a, b) \in A \times B$ i.e., $H(a, b) = \{(h_i, h_j) : h_i \in F(a)$ and $h_j \in G(b)\}$.

The Cartesian product of three or more non-empty vague soft sets can be defined by generalizing the definition of the Cartesian product of two vague soft sets. The Cartesian product $(\tilde{F_1}, A) \times (\tilde{F_2}, A) \times ... \times (\tilde{F_n}, A)$ of the non-empty vague soft sets $(\tilde{F_1}, A)$, $(\tilde{F_2}, A)$, ..., $(\tilde{F_n}, A)$ is the vague soft set of all ordered n-tuples $(h_1, h_2, ..., h_n)$, where $h_i \in \tilde{F_i}(a)$.

Definition 16 [36]**.** Let $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be two vague soft sets over U. Then, the relation between $(\tilde{F}, A)$ and $(\tilde{G}, B)$ is a vague soft subset of $(\tilde{F}, A) \times (\tilde{G}, B)$. The relation between $(\tilde{F}, A)$ and $(\tilde{G}B)$ is of the form $(\tilde{H}_1, S)$, where $S \subset A \times B$ and $\tilde{H}_1(a, b) \forall a, b \in S$. Any subset of $(\tilde{F}, A) \times (\tilde{F}, A)$ is called a relation on $(\tilde{F}, A)$ in parameterized form as follows:

If $(\tilde{F}, A) = \{\tilde{F}(a), \tilde{F}(b), ...\}$, then
$\tilde{F}(a) \Re \tilde{F}(b)$ if and only if $\tilde{F}(a) \times \tilde{F}(b) \in \Re$.

3. Transitive Closure of Vague Soft Set

Suppose that $\Re$ is a relation on a vague soft set (F, A), then $\Re$ may or may not have some property ρ, such as reflexivity, symmetry, or transitivity. If there is a relation S with property ρ containing $\Re$, such that S is a sub-soft set of every relation with property ρ containing $\Re$, then S is called the closure of $\Re$ with respect to ρ. The closure of a relation with respect to a property may or may not exist.

In this section, the concepts of reflexive closure, symmetry closure, and transitive closure of soft sets proposed by Ibrahim et al. [39] are extended to those of vague soft sets, followed by examples to illustrate the operations of the newly defined relations. Then, a novel definition of transitive closure for a vague soft set is proposed along with its properties and an example to illustrate these properties.

3.1 Reflexive Closure

Definition 17 [39]**.** The reflexive closure of R equals $R \cup \Upsilon$, where $\Upsilon = \{(F(a), F(a)) : F(a) \in (F, a)\}$ is the diagonal relation on (F, A).

Note that R is the relation on the soft set. The relation of the vague soft set $\Re$ with respect to its reflexive closure can be formed by adding to $\Re$ all pairs of the form $(F(a), F(a))$ with $F(a)\ in(F, A)$, not already in $\Re$. The addition of these pairs produces a new relation that is reflexive, contains $\Re$, and is contained within any reflexive relation that contains $\Re$. Using $\Re$ instead of R in Definition 13, the following new definition of the reflexive closure of a vague soft set is obtained.

Definition 18. Let $\Re$ be a vague soft set relation on (F, A). The minimal reflexive vague soft set relation containing $\Re$ is called the reflexive closure of $\Re$, denoted by $\bar{r}(\Re)$.

Now, the notion of reflexive closure of a vague soft set is illustrated using the following example.

Example 1. Let $U = \{u_1, u_2, u_3\}$, $A = \{e_1, e_2, e_3\}$. The vague soft set (F, A) is given by

$$F(e_1) = \left\{ \frac{u_1}{\langle 0.1, 0.2 \rangle}, \frac{u_2}{\langle 0.3, 0.7 \rangle}, \frac{u_3}{\langle 0.1, 0.1 \rangle} \right\},$$
$$F(e_2) = \left\{ \frac{u_1}{\langle 0.2, 0.6 \rangle}, \frac{u_2}{\langle 0.1, 0.3 \rangle}, \frac{u_3}{\langle 0.8, 0.9 \rangle} \right\},$$
$$F(e_3) = \left\{ \frac{u_1}{\langle 0.4, 0.6 \rangle}, \frac{u_2}{\langle 0.3, 0.6 \rangle}, \frac{u_3}{\langle 0.3, 0.3 \rangle} \right\}.$$

Consider the vague soft set relation $\Re$ defined on (F, A) as

$$\Re = \left\{ \frac{F(e_1) \times F(e_2)}{\frac{u_1}{\langle 0.1,0.6 \rangle}, \frac{u_2}{\langle 0.1,0.7 \rangle}, \frac{u_3}{\langle 0.3,0.9 \rangle}}, \frac{F(e_2) \times F(e_3)}{\frac{u_1}{\langle 0.2,0.6 \rangle}, \frac{u_2}{\langle 0.1,0.6 \rangle}, \frac{u_3}{\langle 0.3,0.9 \rangle}}, \right.$$
$$\left. \text{and } \frac{F(e_3) \times F(e_3)}{\frac{u_1}{\langle 0.4,0.6 \rangle}, \frac{u_2}{\langle 0.3,0.6 \rangle}, \frac{u_3}{\langle 0.3,0.3 \rangle}} \right\}.$$

Then

$$\begin{aligned} \bar{r}(\Re) =& \Re \cup \{F(e_1) \times F(e_1), F(e_2) \times F(e_2), F(e_3) \times F(e_3)\} \\ =& \{F(e_1) \times F(e_2), F(e_2) \times F(e_3), F(e_3) \times F(e_3)\} \\ & \cup \{F(e_1) \times F(e_1), F(e_2) \times F(e_2), F(e_3) \times F(e_3)\} \\ =& \left\{ \frac{F(e_1) \times F(e_1)}{\frac{u_1}{\langle 0.1,0.2 \rangle}, \frac{u_2}{\langle 0.3,0.7 \rangle}, \frac{u_3}{\langle 0.1,0.1 \rangle}}, \frac{F(e_1) \times F(e_2)}{\frac{u_1}{\langle 0.1,0.6 \rangle}, \frac{u_2}{\langle 0.1,0.7 \rangle}, \frac{u_3}{\langle 0.3,0.9 \rangle}}, \right. \\ & \frac{F(e_2) \times F(e_2)}{\frac{u_1}{\langle 0.2,0.6 \rangle}, \frac{u_2}{\langle 0.1,0.3 \rangle}, \frac{u_3}{\langle 0.8,0.9 \rangle}}, \frac{F(e_2) \times F(e_3)}{\frac{u_1}{\langle 0.2,0.6 \rangle}, \frac{u_2}{\langle 0.1,0.6 \rangle}, \frac{u_3}{\langle 0.3,0.9 \rangle}}, \\ & \left. \text{and } \frac{F(e_3) \times F(e_3)}{\frac{u_1}{\langle 0.4,0.6 \rangle}, \frac{u_2}{\langle 0.3,0.6 \rangle}, \frac{u_3}{\langle 0.3,0.3 \rangle}} \right\}, \end{aligned}$$

3.2 Symmetry Closure

The symmetry closure of a relation $\Re$ is constructed by adding all ordered pairs of the form $(F(b), F(a))$, where $(F(a), F(b))$ is a relation that is not already present in $\Re$. Adding these pairs produces a symmetric relation that contains $\Re$.

Definition 19 [39]**.** The symmetric closure of a relation is obtained by taking the union of a relation with its inverse, i.e., $R \cup R^{-1}$ where $R^{-1} = \{(F(b), F(a)) : (F(a), F(b)) \in R\}$.

The definition of symmetric closure on soft sets by Ibrahim et al. [39] is extended to the symmetric closure of a vague soft set below, followed by an example to illustrate its operation.

Definition 20. The symmetric closure of a relation is obtained by taking the union of the relation with its inverse, i.e., $\Re \cup \Re^{-1}$ where $\Re^{-1} = \{(F(b), F(a)) \colon (F(F(a), F(b)) \in \Re\}$.

In other words, let $\Re$ be a vague soft set relation on (F, A). The minimal symmetric vague soft set relation containing $\Re$ is called the symmetric closure of $\Re$, denoted by $\bar{s}(\Re)$.

Example 2. Consider Example 1

$$\begin{aligned}\bar{s}(\Re) =& \Re \cup \Re^{-1}\\ =& \{F(e_1) \times F(e_2), F(e_2) \times F(e_3), F(e_3) \times F(e_3)\}\\ & \cup \{F(e_2) \times F(e_1), F(e_3) \times F(e_2), F(e_3) \times F(e_3)\}\\ =& \{F(e_1) \times F(e_2), F(e_2) \times F(e_3), F(e_2) \times F(e_1),\\ & F(e_3) \times F(e_2), F(e_3) \times F(e_3)\}.\end{aligned}$$

3.3 Transitive Closure

Now, the definition of transitive closure of a soft set by Ibrahim et al. [39] is extended to the transitive closure of a vague soft set.

The construction of the transitive closure of a relation is more complicated than that of reflexive or symmetric closure. The transitive closure of a relation can be determined by adding new ordered pairs that must be present and then repeating this process until no new ordered pairs are required. R^* is said to be the transitive closure of R, if it satisfies the following conditions:

(i) R^* is transitive.

(ii) $R \subseteq R^*$.

(iii) R^* is the smallest transitive relation containing R.

Definition 21 [37]**.** Let R be a relation on the soft set (F, A). Then, we define $R^* = \bigcup\limits_{i=1}^{\infty} R$.

Definition 22. Let $\Re$ be a relation on a vague soft set (F, A). Then, we define $\Re^* = \bigcup\limits_{i=1}^{\infty} \Re$.

Based on this duuality, the related properties of transitive closure of a vague soft set can also be investigated.

Let $\Re$ be a relation on a vague soft set (F, A) with m elements. Then

(i) transitive $(\Re) = \Re \cup \Re^2 \cup \ldots \cup \Re^m$,

(ii) $M_{\Re^*} = M_\Re \cup M_{\Re^2} \cup \ldots \cup M_{\Re^m}$, where $M_\Re$ is the matrix of the relation $\Re$.

(iii) $M_{\Re_1 \cup \Re_2} = M_{\Re_1} \vee M_{\Re_2}$, where $\Re_1$ and $\Re_2$ are relations on (F, A) with matrices $M_{\Re_1}$ and $M_{\Re_2}$.

The properties of the reflexive closure of a vague soft set can be illustrated by the following example.

Example 3. Suppose that $\Re$ is a relation on (F, A) with $A = \{a_1, a_2, a_3\}$, where $\Re = \{F(a_1) \times F(a_2), F(a_2) \times F(a_3), F(a_3) \times F(a_3)\}$. The zero-one matrix for $\Re$ is given by

$$M_\Re = \begin{bmatrix} \langle 0,1\rangle & \langle 1,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \end{bmatrix}.$$

Thus $M_{\Re^*} = M_\Re \vee M_\Re^2 \vee M_\Re^3$ since $n = 3$.

Now

$$\begin{aligned}\Re^2 &= M_{\Re^2} = M_\Re \cdot M_\Re\\ &= \begin{bmatrix} \langle 0,1\rangle & \langle 1,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \end{bmatrix} \begin{bmatrix} \langle 0,1\rangle & \langle 1,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \end{bmatrix}\\ &= \begin{bmatrix} \langle 0,1\rangle & \langle 1,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \end{bmatrix},\\ \Re^3 &= \Re^2 \cdot \Re = M_{\Re^2} \cdot M_\Re\\ &= \begin{bmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \end{bmatrix} \begin{bmatrix} \langle 0,1\rangle & \langle 1,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \end{bmatrix}\\ &= \begin{bmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \end{bmatrix},\\ M_{\Re^*} &= \begin{bmatrix} \langle 0,1\rangle & \langle 1,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \end{bmatrix} \vee \begin{bmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \end{bmatrix}\\ &\quad \vee \begin{bmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \end{bmatrix}\\ &= \begin{bmatrix} \langle 0,1\rangle & \langle 1,1\rangle & \langle 1,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,1\rangle \end{bmatrix}.\end{aligned}$$

Reading from the zero-one matrix, we see that $\Re^* = \{F(a_1) \times F(a_2), F(a_1) \times F(a_3), F(a_2) \times F(a_3), F(a_3) \times F(a_3)\}$ is the transitive $(\Re)$.

Theorem 1. The relation $\Re$ on a vague soft set (F, A) is transitive if and only if $\Re^n \subseteq \Re$ for every $n \in N$.

Proof. Suppose that $\Re^n \subseteq \Re$ for every $n \in N$. Specifically, $\Re^2 \subseteq \Re$. To prove that $\Re$ is transitive, suppose that $F(a) \times F(b) \in \Re$ and $F(b) \times F(c) \in \Re$; then, by the definition of composition, $F(a) \times F(c) \in \Re^2$. Since $\Re^2 \subseteq \Re$, it is inferred that $F(a) \times F(c)$ is in $\Re$. Therefore, $\Re$ is transitive.

Conversely, suppose that $\Re$ is transitive. It can be proved that $\Re^n \subseteq \Re$ by induction. This is true for $n = 1$. Assume that $\Re^n \subseteq \Re$ for n. Then, we need to show that $\Re^{n+1} \subseteq \Re$. To demonstrate this, assume that $F(a) \times F(b) \in \Re^{n+1}$. Since $\Re^{n+1} = \Re^n \, o \, \Re$, there exists an element $F(x)$ such that $F(a) \times F(x) \in \Re$ and $F(x) \times F(b) \in \Re^n$. Now, $\Re^n \subseteq \Re$ yields $F(x) \times F(b) \in \Re$. Furthermore, as $\Re$ is transitive and $F(a) \times F(x) \in \Re$, it follows that $F(a) \times F(b)$ is in $\Re$. This shows that $\Re^{n+1} \subseteq \Re$, thereby completing the proof. □

Theorem 2. If T and U are two vague soft set functions from (F, A) to (G, B), and $\Re$ and S are two vague soft set functions from (G, B) to (H, C), then $\Re \subset S$ and $T \subset U \Rightarrow \Re \, o \, T \subset S \, o \, U$.

Proof. Suppose that $F(a) \times H(c) \in R \, o \, T$. This implies that there exists $G(b) \in (G, B)$ such that $F(a) \times G(b) \; isin T$ and $G(b) \times H(c)$ is in $\Re$.

Now $\Re \subset S \Rightarrow G(b) \times H(c) \in S$ and $T \subset U \Rightarrow F(a) \times G(b) \in U$. Then $F(a) \times H(c) \in S \, o \, U$, showing that $\Re \, o \, T \subset S \, o \, U$. □

Definition 23. Let $\Re$ be a binary relation on (F, A). The transitive closure of $\Re$ denoted by $\tilde{\Re}$ is the smallest vague soft set relation containing $\Re$ that is transitive.

For the following definitions, lemmas, and proofs, dom is used to denote "domain of."

Definition 24. Let f and g be two vague soft set functions on (F, A) and (G, B), respectively. Then,

(i) f and g are compatible if $f(F(a)) = g(G(a))$ for all $F(a) \in dom f \cap dom g$.

(ii) A set of vague soft set functions Γ is a compatible system of functions if any two functions f and g from Γ are compatible.

Lemma 1.

(a) vague soft set functions f and g are compatible if and only if $f \cup g$ is a function.

(b) vague soft set functions f and g are compatible if and only if $f/((dom f \cap dom g)) = g/((dom f \cap dom g))$.

Proof. The result follows from Definition 20. □

Theorem 3. If Γ is a compatible system of functions, then $\cup\Gamma$ is a function with $dom \cup \Gamma = \cup\{dom f / f \in \Gamma\}$. Then, the function $\cup\Gamma$ extends to all $f \in \Gamma$.

Proof. Clearly $\cup\Gamma$ is a relation. We also prove that this is a function. If $F(a) \times F(b_1) \in \cup\Gamma$ and $F(a) \times F(b_2) \in \cup\Gamma$, then there are functions $f_1, f_2 \in \Gamma$ such that $F(a) \times F(b_1) \in f_1$ and $F(a) \times F(b_2) \in f_2$. However, f_1 and f_2 are compatible and $f_1 \in dom f_1 \cap dom f_2$. So $F(b_1) = f_1(F(a)) = f_2(F(a)) = F(b_2)$.

It is trivial to show that $F(x) \in dom \cup \Gamma$ if and only if $F(x) \in dom$ for some $f \in \Gamma$. □

4. Properties of Closure

In this section, the properties of reflexive closure and symmetric closure of a vague soft set are introduced.

Theorem 4. Let $\Re$ be a vague soft set relation on (F, A). Then

(1) $\bar{r}(\Re) = \Re \cup I$. Therefore, a mapping (called a reflexive closure operator) $\bar{r} : RC\Re(F, A) \to RC\Re(F, A)$ is obtained.

(2) $\bar{s}(\Re) = \Re \cup \Re^{-1}$. Therefore, a mapping (called the symmetric closure operator) $\bar{s} : SC\Re(F, A) \to SC\Re(F, A)$ is obtained.

Proof.

(1) $\Re \subset \Re \cup I$. $\forall a \in A, F(a) \times F(a) \in I \subset \Re \cup I$; therefore, $\Re \cup I$ is reflexive. On the other hand, T has a reflexive vague soft set relation on (F, A) and $\Re \subset T$. By the reflexivity of T, $I \subset T$, if $I \subset T$ and $\Re \subset T$, then $\Re \cup I \subset T$. Therefore, $\bar{r}(\Re) = \Re \cup I$.

(2) $(\Re \cup \Re^{-1})^{-1} = \Re^{-1} \cup (\Re^{-1})^{-1} = \Re^{-1} \cup \Re = \Re \cup \Re^{-1}$, i.e. $\Re \cup \Re^{-1}$ is a symmetric vague soft set relation on (F, A), and $\Re \subset \Re \cup \Re^{-1}$.
If T has a symmetric vague soft set relation on (F, A) and $\Re \subset T$, then $\Re^{-1} \subset T^{-1}$. Since $\Re$ is symmetric iff $\Re = \Re^{-1}$ and $\Re \cup \Re^{-1} \subset T$ then $T = T^{-1}$. Therefore, $\bar{s}(\Re) = \Re \cup \Re^{-1}$. □

Next, some basic properties of the reflexive and symmetric closure operators are proposed.

Theorem 5. The reflexive closure operator $\bar{r}$ has the following properties:

(1) $\bar{r}(M) = M, \bar{r}(I) = I$.

(2) $\forall \Re \in RC\Re(F, A), \Re \subset \bar{r}(\Re)$.

(3) $\forall \Re, Q \in RC\Re(F, A), \bar{r}(\Re \cup Q) = \bar{r}\Re \cup \bar{r}Q, \bar{r}(\Re \cap Q) = \bar{r}\Re \cap \bar{r}Q$.

(4) $\forall \Re, Q \in RC\Re(F, A)$, if $\Re \subset Q$, then $\bar{r}(\Re) \subset \bar{r}(Q)$.

(5) $\forall \Re \in RC\Re(F, A), \bar{r}(\bar{r}(\Re)) = \bar{r}(\Re)$.

Proof.

(1) By the reflexivity of M and I, $\bar{r}(M) = M, \bar{r}(I) = I$.

(2) $\forall \Re \in RC\Re(F, A)$, by Theorem 4 (1) and $\bar{r}(\Re) = \Re \cup I$, $\bar{r}(\Re) = \Re \cup I \supset \Re$.

(3) $\forall \Re, Q \in RC\Re(F, A)$, by Theorem 4, $\bar{r}(\Re \cup Q) = (\Re \cup Q) \cup I = (\Re \cup I) \cup (Q \cup I) = \bar{r}(\Re) \cup \bar{r}(Q)$. $\bar{r}(\Re \cap Q) = (\Re \cap Q) \cup I = (\Re \cap I) \cap (Q \cap I) = \bar{r}(\Re) \cap \bar{r}(Q)$.

(4) $\forall \Re, Q \in RC\Re(F, A), \Re \subset Q$ by (3) and if $\Re \subset Q$, then $\Re \cup Q = Q$ and $\Re \cap Q = \Re$, $\bar{r}(Q) = \bar{r}(\Re \cup Q) = \bar{r}(\Re) \cup \bar{r}(Q) \supset \bar{r}(\Re)$.

(5) $\forall \Re \in RC\Re(F, A)$, by Theorem 4 (1), $\bar{r}(\Re) = \Re \cup I$. Hence $\bar{r}(\bar{r}(\Re)) = \bar{r}(\Re \cup I) = (\Re \cup I) \cup I = \Re \cup I = \bar{r}(\Re)$. □

Theorem 6. The symmetric closure operator $\bar{s}$ has the following properties:

(1) $\bar{s}(M) = M, \bar{s}(I) = I$.

(2) $\forall \Re \in SC\Re(F, A), \Re \subset \bar{s}(\Re)$.

(3) $\forall \Re, Q \in SC\Re(F, A), \bar{s}(\Re \cup Q) = \bar{s}\Re \cup \bar{s}Q, \bar{s}(\Re \cap Q) = \bar{s}\Re \cap \bar{s}Q$.

(4) $\forall \Re, Q \in SC\Re(F, A)$, if $\Re \subset Q$, then $\bar{s}(\Re) \subset \bar{s}(Q)$.

(5) $\forall \Re \in SC\Re(F, A), \bar{s}(\bar{s}(\Re)) = \bar{s}(\Re)$.

Proof.

(1) By symmetry of m, M, and I, $\bar{s}(m) = m, \bar{s}(M) = M, \bar{s}(I) = I$.

(2) $\forall \Re \in SC\Re(F, A)$, by Theorem 5(2), $\Re \subset \bar{s}(\Re)$.

(3) $\forall \Re, Q \in SC\Re(F, A)$, by Theorem 5 and $(\Re \cup Q)^{-1} = \Re^{-1} \cup Q^{-1}$ and $(\Re \cap Q)^{-1} = \Re^{-1} \cap Q^{-1}$, we have

$$
\begin{aligned}
\bar{s}(\Re \cup Q) &= (\Re \cup Q) \cup (\Re \cup Q)^{-1} \\
&= (\Re \cup Q) \cup (\Re^{-1} \cup Q^{-1}) \\
&= (\Re \cup \Re^{-1}) \cup (Q \cup Q^{-1})^{-1} \\
&= \bar{s}(\Re) \cup \bar{s}(Q),
\end{aligned}
$$

and for $\bar{s}(\Re \cap Q) = \bar{s}\Re \cap \bar{s}Q$, we have

$$
\begin{aligned}
\bar{s}(\Re \cap Q) &= (\Re \cap Q) \cap (\Re \cap Q)^{-1} \\
&= (\Re \cap Q) \cap (\Re^{-1} \cap Q^{-1}) \\
&= (\Re \cap \Re^{-1}) \cap (Q \cap Q^{-1})^{-1} \\
&= \bar{s}(\Re) \cap \bar{s}(Q).
\end{aligned}
$$

(4) $\forall \Re, Q \in SC\Re(F, A)$, $\Re \subset Q$, by (3) and Theorem 5(4), $\bar{s}(Q) = \bar{s}(\Re \cup Q) = \bar{s}(\Re) \cup \bar{s}(Q) \supset \bar{s}(\Re)$.

(5) $\forall \Re \in SC\Re(F, A)$, by Theorem 5(2), $\bar{s}(\Re) = \Re \cup \Re^{-1}$. Hence

$$
\begin{aligned}
\bar{s}(\bar{s}(\Re)) &= \bar{s}(\Re \cup \Re^{-1}) = (\Re \cup \Re^{-1}) \cup (\Re \cup \Re^{-1})^{-1} \\
&= (\Re \cup \Re^{-1}) \cup (\Re^{-1} \cup (\Re^{-1})^{-1}) \\
&= (\Re \cup \Re^{-1}) \cup (\Re^{-1} \cup \Re) \\
&= (\Re \cup \Re^{-1}) \\
&= \bar{s}(\Re).
\end{aligned}
$$

□

5. Conclusion

Reflexivity, symmetry, and transitivity are three of the most important properties of vague soft set relations. This work has shown how reflexive, symmetric, and transitive closure of vague soft set relations can be determined. After establishing some properties of the transitive closure of a vague soft set, some properties of the symmetric closure and reflexive closure operators were provided on a vague soft set. To extend this work, applications of transitive closure of vague soft sets in decision-making can be considered.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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