Separation Axioms in Mixed Fuzzy Topological Spaces

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Abstract

Herein, we define fuzzy T_0 -space, fuzzy T_1 -space, fuzzy T_2 (or Hausdorff), as well as fuzzy regular and fuzzy normal spaces in mixed fuzzy topological spaces, and then establish relationships among these spaces. We provide some results for the abovementioned spaces in mixed fuzzy topological spaces.

Keywords: Fuzzy topological spaces, Mixed fuzzy topology, Separation Axioms, T_0 -space, T_1 -space, T_2 (or Hausdorff) spaces

1. Introduction

The concept of mixing two topologies to obtain a new topology is not a new concept. In 1958, Alexiewicz and Semadeni [1] introduced and investigated a mixed topology on a set via norm spaces. Thereafter, many mathematicians worldwide investigated mixed topologies and discovered some interesting and applicable results regarding mixed topologies. In this regard, some remarkable results have been reported by Cooper [2], Buck [3], Wiweger [4], and many others. A new era began after the inception of the fuzzy set theory in 1965 by Zadeh [5]. Since the inception of the notion of fuzzy sets and fuzzy logic, fuzziness has been applied to studies in almost all branches of science and technology. The notion of fuzziness has been applied to topology for the first time by Chang [6], and fuzzy topological spaces have been introduced and then investigated extensively. The concept of strong separation and strong countability in fuzzy topological spaces was introduced and investigated by Stadler and de Prada Vicente [7]. In 1995, the concept of mixed fuzzy topological spaces was investigated from different perspectives by Das and Baishya [8]. Recently, in 2012, Tripathy and Ray [9] generalized the definition of mixed fuzzy topology introduced by Das and Baishya [8], by replacing fuzzy points with fuzzy sets. Tripathy and Ray [9] introduced the concept of countability base on a mixed fuzzy topology and proved the existence of three types of countability. However, this concept of mixed fuzzy topology is not a generalization of the concept of a crisp mixed topology (or simply mixed topology) by Wiweger [4], Alexiewicz and Semadeni [1], Cooper [2], and others. Ghanim et. al [10] investigated separation axioms. Recently, W. F. Al-Omeri [11] investigated a mixed b-fuzzy topology using the definition by Das and Baishya [8].

Most recently, Al-Shami and his colleagues [12–21] investigated separation axioms in topological spaces and obtained some important findings. Rashid and Ali [22] introduced separation axioms in a mixed fuzzy topology using the concept of mixed fuzzy topology defined by Das and Baishya [8]. Recently, Tripathy and Ray [9] generalized the definition of mixed fuzzy topology by Das and Baishya [8]. In this study, we redefined and investigated separation axioms in mixed

Received: Jan. 3, 2021
 Revised : Sep. 12, 2021
 Accepted: Sep. 15, 2021

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fuzzy topological spaces introduced by Tripathy and Ray [9]. We investigated a few properties of separation axioms that varied slightly, as introduced by Rashid and Ali [22]. We have previously proven some results and constructed a few concrete numerical examples of T_0 , T_1 , T_2 (Hausdorff), and regular and normal mixed fuzzy topological spaces such that the spaces can be visualized more effectively. Herein, we provide the basic definitions in Section 2, although most of the concepts, definitions, and notations used herein are standard ones.

2. Preliminaries

Definition 1 [23]. Let X be a non-empty set, and I be the unit interval $[0, 1]$. A fuzzy set A in X is characterized by a function $\mu_A : X \rightarrow I$, where μ_A is known as the membership function of A . $\mu_A(x)$ representing the membership grade of x in A . The empty fuzzy set is defined as $\mu_\phi(x) = 0$ for all $x \in X$. Additionally, X can be regarded as a fuzzy set in itself, defined $\mu_X(t) = 1$ for all $t \in X$. Furthermore, an ordinary subset A of X can be regarded as a fuzzy set in X if its membership function is regarded as the usual characteristic function of A , i.e., $\mu_A(x) = 1$, for all $x \in X$ and $\mu_A(x) = 0$ for all $x \in X - A$. The two fuzzy sets A and B are equal if $\mu_A = \mu_B$. A fuzzy set A is contained in a fuzzy set B , written as $A \subseteq B$, if $\mu_A \leq \mu_B$. The complement of a fuzzy set A in X is a fuzzy set A in X , defined as $\mu_{A^c} = 1 - \mu_A$. We write $A^c = coA$ to avoid confusion.

The union and intersection of a set $\{A_i : i \in I\}$ of fuzzy sets in X to be written as $\bigcup_{i=1} A_i$ and $\bigcap_{i=1} A_i$, separately, are defined as follows:

$$\mu_{\bigcup_{i \in I} A_i}(x) = \sup\{\mu_{A_i}(x) : i \in I\} \text{ for all } x \in X,$$

and

$$\mu_{\bigcap_{i \in I} A_i}(x) = \inf\{\mu_{A_i}(x) : i \in I\} \text{ for all } x \in X.$$

Definition 2 [6]. Let $I = [0, 1]$, X be a non-empty set, and I^X be a set of all mappings from X to I , i.e., the class of all fuzzy sets in X .

A fuzzy topology on X is a family τ of member I^X , such that

- (i) $\bar{1}, \bar{0} \in \tau$.
- (ii) For any finite subset $\mathfrak{B} = \{B_i\}_{i=1}^n$ of members of τ , $\bigcap_{i=1}^n \{B_i\} \in \tau$.
- (iii) For any arbitrary set Δ of members of τ , $\bigcup_{B \in \Delta} \{B\} \in \tau$.

The pair (X, τ) is known as a fuzzy topological space (fts), and the members of τ are known as τ -open fuzzy sets.

Definition 3 [24]. The closure of a fuzzy set A in an fts (X, τ) is defined as the intersection of all closed supersets of A , i.e.,

$$\bar{A} = \bigcap \{F : A \subset F, F \text{ is a closed fuzzy set}\}.$$

Definition 4 [23]. A fuzzy set in X is known as a fuzzy point if and only if it assumes the value 0 for all $y \in X$ except one, e.g., $x \in X$. If its value at x is α ($0 < \alpha \leq 1$), then this point is denoted by x_α , and we refer to point x as its support.

Definition 5 [23]. A fuzzy point x_α is contained in a fuzzy set A or belongs to A , denoted by $x_\alpha \in A$, if and only if $\alpha \leq A(x)$.

Definition 6 [23]. Two fuzzy sets A and B in X are intersecting if a point $x \in X$ exists such that $(A \cap B)(x) \neq 0$. In this case, A and B intersect at x .

Definition 7 [23]. A fuzzy set A in a fts (X, τ) is known as the neighborhood of a fuzzy point $x_\alpha \in X$ if and only if $B \in \tau$ exists such that $x_\alpha \in B \subseteq A$; a neighborhood A is known as an open neighborhood if and only if A is fuzzy open. A family comprising all the neighborhoods of x_α is known as a system of neighborhoods of x_α .

Definition 8 [23]. A fuzzy point x_α is quasi-coincident with a fuzzy set A , denoted by $x_\alpha q A$, if and only if $\alpha + A(x) > 1$ or $\alpha > (A(x))^c$.

Definition 9 [23]. A fuzzy set A is quasi-coincident with B and is denoted by $A q B$ if and only if $x \in X$ exists such that $A(x) + B(x) > 1$.

It is clear that if A and B are quasi-coincident at x , then both $A(x)$ and $B(x)$ are non-zero at x ; hence, A and B intersect at x .

Definition 10 [23]. A fuzzy set A in an fts (X, τ) is known as a quasi-neighborhood (Q -neighborhood) of x_α if and only if $\exists A_1 \in \tau$, such that $A_1 \subseteq A$ and $x_\alpha q A_1$. A family U_{x_α} comprising all the Q -neighborhoods of x is known as the Q -neighborhood system of x . The intersection of two Q -neighborhoods of x_α is a quasi-neighborhood.

Theorem 1 [8]. Let (X, τ_1) and (X, τ_2) be two fuzzy topological spaces. Consider fuzzy sets $\tau_1(\tau_2) = \{A \in I^X : \text{For any } x q A, \text{ a } \tau_2\text{-}Q\text{-neighborhood } A \text{ of } x \text{ exists such that } \tau_1\text{-closure } \bar{A}|_\alpha \subseteq A\}$. Subsequently, this family of fuzzy sets will form a fuzzy topology on X , and this topology is known as a mixed fuzzy topology on X .

Definition 11 [10]. A fts (X, τ) is known as a fuzzy T_0 -space if and only if for any pair of fuzzy points x_α and y_β in

I^X with $x \neq y$, open fuzzy sets U and V exist in τ such that $x_\alpha \in U \subseteq co(y_\beta)$ or $y_\beta \in V \subseteq co(x_\alpha)$ in τ .

Definition 12 [10]. A fts (X, τ) is known as a fuzzy T_1 -space if and only if for any pair of fuzzy points x_α and y_β in I^X with $x \neq y$, open fuzzy sets U and V exist in τ such that $x_\alpha \in U$ and $y_\beta \notin U$, and $x_\alpha \notin V$ and $y_\beta \in V$.

Definition 13 [10]. A fts (X, τ) is known as the fuzzy Hausdorff or fuzzy T_2 -space if and only if for any pair of fuzzy points x_α and y_β in I^X with $x \neq y$, open fuzzy sets U and V exist such that $x_\alpha \in U$ and $y_\beta \in V$, and $U \cap V = \bar{0}$.

Definition 14 [10]. A fts (X, τ) is known as fuzzy regular if and only if for any fuzzy point x and a closed fuzzy set B with $x \in B^c$, open fuzzy sets U and V exist in τ such that $x_\alpha \in U$ and $B \subseteq V$, and $U \subseteq 1 - V$.

Proposition 1 [25]. fts (X, τ) is known as a fuzzy regular space if and only if for any fuzzy point x_α in I^X and $A \in \tau$ with $\alpha < A(x)$, $B \in \tau$ exists such that $\alpha < B(x)$ and $\bar{B} \subseteq A$.

Definition 15 [22]. A fts (X, τ) is known as a fuzzy normal space if and only if for each closed set B and an open fuzzy set U in (X, τ) where $B \subseteq U$, $V \in \tau$ exists such that $B \subseteq V \subseteq \bar{V} \subseteq U$.

Lemma 1 [8]. Let τ_1 and τ_2 be two fuzzy topological spaces on set X . If every τ_1 - Q -neighbourhood of x is a τ_2 - Q -neighbourhood of x for all fuzzy points x , then τ_1 is coarser than τ_2 .

Theorem 2 [9]. Let (X, τ_1) and (X, τ_2) be two fuzzy topological spaces. Consider fuzzy sets $\tau_1(\tau_2) = \{A \in I^X : \text{For any fuzzy set } B \text{ in } X \text{ with } AqB, \text{ a } \tau_2\text{-open set } A_1 \text{ exists such that } A_1qB \text{ and } \tau_1\text{-closure } \bar{A}_1 \subseteq A\}$. Subsequently, this family of fuzzy sets will form a fuzzy topology on X , and this topology is known as a mixed fuzzy topology on X .

Theorem 3 [8]. Let τ_1 and τ_2 be two fuzzy topological spaces on set X . Subsequently, the mixed fuzzy topology $\tau_1(\tau_2)$ is coarser than τ_2 . This is represented as $\tau_1(\tau_2) \subseteq \tau_2$.

Proof. Because every fuzzy point is also a fuzzy singleton set, the theorem is valid with respect to the definition in Theorem 2. $\square$

3. Results

In this section, we analogously define the basic definition of T_0, T_1, T_2 (or Hausdorff), as well as regular and normal mixed fuzzy topological spaces $(X, \tau_1(\tau_2))$, and then provide some results in mixed fuzzy topological spaces. The definitions used herein are as follows:

Definition 16. A mixed fts $(X, \tau_1(\tau_2))$ is known as a fuzzy

T_0 -space if and only if for any pair of fuzzy points x_α and y_β in I^X with $x \neq y$, open fuzzy sets U and V exist in $\tau_1(\tau_2)$ such that $x_\alpha \in U \subseteq co(y_\beta)$ or $y_\beta \in V \subseteq co(x_\alpha)$ in $\tau_1(\tau_2)$.

Definition 17. A mixed fts $(X, \tau_1(\tau_2))$ is known as a mixed fuzzy T_1 -space if and only if for any pair of fuzzy points x_α and y_β in I^X with $x \neq y$, open fuzzy sets U and V in $\tau_1(\tau_2)$ exist such that $x_\alpha \in U$ and $y_\beta \notin U$ as well as $x_\alpha \notin V$ and $y_\beta \in V$.

Definition 18. A mixed fts $(X, \tau_1(\tau_2))$ is known as a fuzzy Hausdorff or fuzzy T_2 -space if and only if for any pair of fuzzy points x_α and y_β in I^X with $x \neq y$, open fuzzy sets U and V exist in $\tau_1(\tau_2)$ such that $x_\alpha \in U$ and $y_\beta \in V$, and $U \cap V = \bar{0}$.

Definition 19. A mixed fts $(X, \tau_1(\tau_2))$ is known as a fuzzy regular space if and only if for any fuzzy point x_α in I^X and a closed set B in $\tau_1(\tau_2)$ with $x \in B^c$, open fuzzy sets U and V exist in $\tau_1(\tau_2)$ such that $x_\alpha \in U$ and $B \subseteq V$, and $U \subseteq 1 - V$.

Definition 20. A mixed fts $(X, \tau_1(\tau_2))$ is known as a fuzzy normal space if and only if for each closed fuzzy set B and an open fuzzy set U in $\tau_1(\tau_2)$ where $B \subseteq U$, $V \in \tau_1(\tau_2)$ exists such that $B \subseteq V \subseteq \bar{V} \subseteq U$.

Example1. Let us consider the following example of mixed fuzzy topology, which comprises fuzz T_0 , fuzzy T_1 and fuzzy T_2 .

Let $X = \{x, y\}$ and $x \neq y$; therefore

$$\tau_1 = \{\bar{0}, \bar{1}, \{(x, \alpha), (y, 0)\}, \{(x, 0), (y, \alpha)\}, \{(x, \alpha), (y, \beta)\} \mid \alpha, \beta \in [0.6, 1]\},$$

and

$$\begin{aligned} \tau_2 = \{ & \bar{0}, \bar{1}, \{(x, 0.3), (y, 0)\}, \{(x, 0), (y, 0.3)\}, \\ & \{(x, 0.5), (y, 0)\}, \{(x, 0), (y, 0.5)\}, \\ & \{(x, 0.3), (y, 0.3)\}, \{(x, 0.5), (y, 0.5)\}, \\ & \{(x, 1), (y, 0)\}, \{(x, 0), (y, 1)\} \} \end{aligned}$$

are fuzzy Hausdorff topological spaces. We constructed $\tau_1(\tau_2)$ from these topologies. In addition, $\tau_1(\tau_2) \subseteq \tau_2$. Therefore we verified the belongingness of $\{(x, 0.3), (y, 0.3)\}, \{(x, 0.5), (y, 0.5)\}, \{(x, 0.3), (y, 0)\}, \{(x, 0), (y, 0.3)\}, \{(x, 0.5), (y, 0)\}, \{(x, 0), (y, 0.5)\}, \{(x, 1), (y, 0)\}$, and $\{(x, 0), (y, 1)\}$ in $\tau_1(\tau_2)$. As an example, $\{(x, 0.3), (y, 0.3)\} \in \tau_2$ for $A_1 = \{(x, \delta) : \delta > 0.7\}$; AqA_1 and the τ_1 closure of $(A = \{(x, 0.3), (y, 0.3)\}) = A \subseteq A$. Because $\{(x, 0.7), (y, 0.7)\}$ is an open set in τ_1 , $\{(x, 0.3), (y, 0.3)\}$ is a closed set in τ_1 . (Similarly, for $A_2 = \{(y, \delta) : \delta > 0.7\}$ or $A_3 = \{(x, \gamma), (y, \delta) \mid \gamma, \delta \in (0.7, 1]\}$; AqA_2 and AqA_3 are also a τ_1 closure of

$$(A = \{(x, 0.3), (y, 0.3)\}) = A \subseteq A.)$$

Therefore $A \in \tau_1(\tau_2)$. However, for $B = \{(x, 0.5), (y, 0.5)\} \in \tau_2$ and $A_2 = \{(x, \delta) : \delta > 0.5\} BqA_2$ and the τ_1 closure of $(B = \{(x, 0.5), (y, 0.5)\}) = \bar{1} \notin B$.

Therefore, $B \notin \tau_1(\tau_2)$. Similarly, as in the case of $A = \{(x, 0.3), (y, 0.3)\} \in \tau_2$, $\{(x, 1), (y, 0)\}$ and $\{(x, 0), (y, 1)\}$ will be members of $\tau_1(\tau_2)$. Furthermore, τ_1 is the closure of $\{(x, 1), (y, 0)\} = \{(x, 1), (y, 0)\}$, as $\{(x, 0), (y, 1)\}$ is an open set in τ_1 ; therefore, the complement $\{(x, 1), (y, 0)\}$ is closed. In addition, the τ_1 closure of $\{(x, 0), (y, 1)\} = \{(x, 0), (y, 1)\}$ as $\{(x, 1), (y, 0)\}$ is an open set in τ_1 .

The complement of $\{(x, 0.3), (y, 0)\}$ in τ_1 is $\{(x, 0.7), (y, 1)\}$, which is an open set such that $\{(x, 0.3), (y, 0)\}$ is a closed set in τ_1 . Therefore, as in the case of $A = \{(x, 0.3), (y, 0)\} \in \tau_1(\tau_2)$, we have $\{(x, 0), (y, 0.3)\} \in \tau_1(\tau_2)$.

However, the complement of $\{(x, 0.5), (y, 0)\}$ in τ_1 is $\{(x, 0), (y, 0.5)\}$ and not fuzzy open in τ_1 ; therefore, $\{(x, 0.5), (y, 0)\}$ is not fuzzy closed, and the τ_1 -closure of $\{(x, 0.5), (y, 0)\} \not\subseteq \{(x, 0.5), (y, 0)\}$.

Similarly, $\{(x, 0.5), (y, 0)\} \notin \tau_1(\tau_2)$ and $\{(x, 0), (y, 0.5)\} \notin \tau_1(\tau_2)$.

Therefore, $\tau_1(\tau_2) = \{\bar{0}, \bar{1}, \{(x, 0.3), (y, 0.3)\}, \{(x, 0.3), (y, 0)\}, \{(x, 0), (y, 0.3)\}, \{(x, 1), (y, 0)\}, \{(x, 0), (y, 1)\}\}$.

Next, we prove that $\tau_1(\tau_2)$ is fuzzy T_0 , fuzzy- T_1 , and fuzzy- T_2 .

For x_α and y_β in I^X , $x_\alpha \in \{(x, 1), (y, 0)\} \subseteq co(y_\beta)$ and $y_\beta \in \{(x, 0), (y, 1)\} \subseteq co(x_\alpha)$.

Additionally, $\{(x, 1), (y, 0)\} \cap \{(x, 0), (y, 1)\} = \bar{0}$.

Therefore, $\tau_1(\tau_2)$ qualifies as fuzzy- T_0 , fuzzy- T_1 , and fuzzy- T_2 .

Example 2. Let $X = \{x, y\}$; subsequently, $\tau_1 = \{\bar{1}, \bar{0}, \{(x, 0), (y, 1)\}\}$ and $\tau_2 = \{\bar{1}, \bar{0}, \{(x, 1), (y, 0)\}, \{(x, 0.7), (y, 0.7)\}, \{(x, 0.7), (y, 0)\}, \{(x, 1), (y, 0.7)\}\}$ are fuzzy topologies. Therefore, $\tau_1(\tau_2) = \{\bar{1}, \bar{0}, \{(x, 1), (y, 0)\}\}$, since $\{(x, 1), (y, 0)\}$ is a closed fuzzy set in τ_1 . For any fuzzy set B in X with $\{(x, 1), (y, 0)\}qB$, we have a τ_2 -open fuzzy set $\{(x, 1), (y, 0)\}$ such that $\{(x, 1), (y, 0)\}qB$ and τ_1 -closure $(\{(x, 1), (y, 0)\}) \subseteq \{(x, 1), (y, 0)\}$.

Clearly, $\tau_1(\tau_2)$ is mixed fuzzy T_0 but not mixed fuzzy T_1 , as $x_\alpha \in \{(x, 1), (y, 0)\} \subseteq co(y_\beta)$ and $\nexists A \in \tau_1(\tau_2)$ such that $y_\alpha \in A \subseteq co(x_\beta)$.

Next, we consider a mixed fuzzy topological space that is fuzzy T_1 but not fuzzy T_2 .

Example 3. Let $X = \{x, y, z\}$; subsequently, $\tau_1 = \{\bar{1}, \bar{0}, \{(x, 1), (y, 1)(z, 0)\}, \{(x, 1), (y, 0)(z, 1)\}, \{(x, 0), (y, 1)(z, 1)\}, \{(x, 1), (y, 0)(z, 0)\}, \{(x, 0), (y, 1)(z, 0)\}, \{(x, 0), (y,$

$0)(z, 1)\}, \{(x, 0.7), (y, 0)(z, 0)\}, \{(x, 0.7), (y, 1)(z, 0)\}, \{(x, 0.7), (y, 0)(z, 1)\}, \{(x, 0.7), (y, 1)(z, 1)\}\}$ and $\tau_2 = \{\bar{1}, \bar{0}, \{(x, 1), (y, 1)(z, 0)\}, \{(x, 1), (y, 0)(z, 1)\}, \{(x, 0), (y, 1)(z, 1)\}, \{(x, 1)(y, 0)(z, 0)\}, \{(x, 0), (y, 1)(z, 0)\}, \{(x, 0), (y, 0)(z, 1)\}\}$ are fuzzy topologies.

Therefore, $\tau_1(\tau_2) = \{\bar{1}, \bar{0}, \{(x, 1), (y, 1)(z, 0)\}, \{(x, 1), (y, 0)(z, 1)\}, \{(x, 0), (y, 1)(z, 1)\}, \{(x, 1), (y, 0)(z, 0)\}, \{(x, 0), (y, 1)(z, 0)\}, \{(x, 0), (y, 0)(z, 1)\}\}$.

Because every open fuzzy set in τ_2 is a closed fuzzy set in τ_1 . Therefore, every open fuzzy set in τ_2 is an open fuzzy set in $\tau_1(\tau_2)$, as shown in Example 2.

Next, we demonstrate that for every pair $x_\alpha, y_\beta, y_\alpha, z_\beta$, or z_α, x_β , open fuzzy sets U and V exist in $\tau_1(\tau_2)$ such that $x_\alpha \in U \subseteq co(y_\beta)$ and $y_\beta \in V \subseteq co(x_\alpha)$. We will verify this only for the pair x_α, y_β ; similarly, other cases can be completed. We have open fuzzy sets $\{(x, 1), (y, 0)(z, 1)\}$ and $\{(x, 0), (y, 1)(z, 1)\}$ in $\tau_1(\tau_2)$ such that $x_\alpha \in \{(x, 1), (y, 0)(z, 1)\} \subseteq co(y_\beta)$ and $y_\beta \in \{(x, 0), (y, 1)(z, 1)\} \subseteq co(x_\alpha)$.

Therefore, $\tau_1(\tau_2)$ is fuzzy T_1 .

However, $\{(x, 1), (y, 0)(z, 1)\} \not\subseteq co(\{(x, 0), (y, 1)(z, 1)\})$.

Therefore, $\tau_1(\tau_2)$ is not a mixed fuzzy T_2 space.

Next, we establish the following theorems:

Theorem 4. If $\tau_1 \subseteq \tau_2$ and τ_1 is fuzzy regular, then $\tau_1 \subseteq \tau_1(\tau_2)$.

Proof. Let A be a τ_1 - Q -neighborhood of x_α , then $\exists A_1 \in \tau_1$ such that

$$\begin{aligned} x_\alpha q A_1 &\subseteq A \\ \Rightarrow \alpha + A_1(x) &> 1 \quad (0 < \alpha \leq 1) \\ \Rightarrow A_1(x) &> 1 - \alpha = \beta \quad (\text{say}) \\ \Rightarrow A_1(x) &> \beta, \text{ where } 0 < \beta \leq 1 \\ \Rightarrow B \in \tau_1 &\text{ exists such that } \beta < B(x) \text{ and } \tau_1\text{-closure of} \\ &B \subseteq A_1 \\ &(\text{Since } \tau_1 \text{ is fuzzy regular}). \end{aligned}$$

Therefore,

$$\begin{aligned} 1 - \alpha &< B(x) \\ \Rightarrow 1 &< B(x) + \alpha \text{ and } \tau_1\text{-closure } \bar{B} \subseteq A_1 \\ \Rightarrow x_\alpha q B, &\text{ and } B \in \tau_2 \text{ and } \tau_1\text{-closure } \bar{B} \subseteq A_1 \subseteq A \\ \Rightarrow A &\in \tau_1(\tau_2) \quad (\text{By definition of } \tau_1(\tau_2)) \\ \Rightarrow A &\text{ is a } \tau_1(\tau_2) \text{ quasi-neighborhood of } x_\alpha \\ \Rightarrow \tau_1 &\subseteq \tau_1(\tau_2). \end{aligned}$$

Hence, the proof is completed. $\square$

Theorem 5. If $\tau_1 \subseteq \tau_2$ and τ_1 are fuzzy regular, then $\tau_1(\tau_2)$ is a fuzzy- T_0 , fuzzy- T_1 , and fuzzy- T_2 (or Hausdorff) space.

Proof. If τ_1 is fuzzy regular, then τ_1 is a fuzzy- T_0 , fuzzy- T_1 , and fuzzy- T_2 (or Hausdorff) space.

Furthermore, $\tau_1 \subseteq \tau_1(\tau_2)$ (by Theorem 4).

1) We prove that $\tau_1(\tau_2)$ is a fuzzy T_0 -space.

Let x_α, y_β be any two fuzzy points such that $x \neq y$. Because τ_1 is a fuzzy T_0 -space, without loss of generality, let an open fuzzy set $U \in \tau_1$ such that

$$\begin{aligned} x_\alpha \in U &\subseteq co(y_\beta) \\ \Rightarrow x_\alpha \in U \in \tau_1(\tau_2) \text{ and } y_\beta \notin U &[\text{since } \tau_1 \subseteq \tau_1(\tau_2)] \\ \Rightarrow \exists \text{ open fuzzy set } U \in \tau_1(\tau_2) \text{ such that } x_\alpha \in U &\subseteq co(y_\beta) \\ \Rightarrow \tau_1(\tau_2) \text{ is fuzzy } T_0 \text{ space.} \end{aligned}$$

2) Next, we prove that $\tau_1(\tau_2)$ is a fuzzy T_1 -space.

Let x_α and y_β be any two fuzzy points such that $x \neq y$. Because τ_1 is a fuzzy T_1 -space, open fuzzy sets V_1 and V_2 in τ_1 exist such that $x_\alpha \in V_1$ and $y_\beta \notin V_1$, and $x_\alpha \notin V_2$ and $y_\beta \in V_2$. Since $\tau_1 \subseteq \tau_1(\tau_2)$, therefore V_1 and V_2 are in $\tau_1(\tau_2)$ such that $x_\alpha \in V_1$ and $y_\beta \notin V_1$, and $x_\alpha \notin V_2$ and $y_\beta \in V_2$. Therefore, $\tau_1(\tau_2)$ is a fuzzy T_1 -pace.

3) Finally, we prove that $\tau_1(\tau_2)$ is a fuzzy Hausdorff space.

Let x_α and y_β be any two fuzzy points such that $x \neq y$.

$\Rightarrow V_1$ and V_2 exist in τ_1 such that $x_\alpha \in V_1$ and $y_\beta \in V_2$ and $V_1 \cap V_2 = \bar{0}$ (since τ_1 is fuzzy Hausdorff).

$\Rightarrow V_1$ and V_2 are in $\tau_1(\tau_2)$ such that $x_\alpha \in V_1$ and $y_\beta \in V_2$, and $V_1 \cap V_2 = \bar{0}$. [since $\tau_1 \subseteq \tau_1(\tau_2)$]

$\Rightarrow \tau_1(\tau_2)$ is a fuzzy Hausdorff space. $\square$

Theorem 6. If $\tau_1(\tau_2)$ is a fuzzy T_0 -space, then τ_2 is also a fuzzy T_0 -space.

Proof. We assume $\tau_1(\tau_2)$ to be a fuzzy T_0 -space.

Let x_α and y_β be any two fuzzy points such that $x \neq y$.

Without loss of generality, let an open fuzzy set $U \in \tau_1(\tau_2)$ exist such that $x_\alpha \in U \subseteq co(y_\beta)$.

Since $\tau_1(\tau_2) \subseteq \tau_2$, $U \in \tau_2$ such that $x_\alpha \in U \subseteq co(y_\beta)$.

Hence, τ_2 is a fuzzy T_0 -space.

Theorem 7. If $\tau_1(\tau_2)$ is a fuzzy T_1 -space, then τ_2 is also a fuzzy T_1 -space.

Proof. Let us assume that $\tau_1(\tau_2)$ is a fuzzy T_1 -space.

Let x_α and y_β be any two fuzzy points such that $x \neq y$. Because $\tau_1(\tau_2)$ is a fuzzy T_1 -space, open fuzzy sets V_1 and V_2 exist in $\tau_1(\tau_2)$ such that $x_\alpha \in V_1$ and $y_\beta \notin V_1$, $x_\alpha \notin V_2$, and $y_\beta \in V_2$. Since $\tau_1(\tau_2) \subseteq \tau_2$, V_1 and V_2 exist in τ_2 such that $x_\alpha \in V_1$ and $y_\beta \notin V_1$, and $x_\alpha \notin V_2$ and $y_\beta \in V_2$.

Hence, τ_2 is also a fuzzy T_1 -space. $\square$

Theorem 8. If $\tau_1(\tau_2)$ is a fuzzy T_2 -space (or Hausdorff), then τ_2 is also a fuzzy T_2 -space.

Proof. Let x_α and y_β be any two fuzzy points such that $x \neq y$.

Because $\tau_1(\tau_2)$ is fuzzy Hausdorff, V_1 and V_2 exist in $\tau_1(\tau_2)$ such that $x_\alpha \in V_1$ and $y_\beta \in V_2$, and $V_1 \cap V_2 = \bar{0}$.

Hence, V_1 and V_2 exist in τ_2 such thauto-at let $x_\alpha \in V_1$ and $y_\beta \in V_2$, and $V_1 \cap V_2 = \bar{0}$ (since $\tau_1(\tau_2) \subseteq \tau_2$).

Hence, τ_2 is also a fuzzy T_2 -space. $\square$

Theorem 9. Let $\tau_1 \supseteq \tau_2$ and τ_2 be fuzzy regular; therefore, $\tau_1(\tau_2)$ is also fuzzy regular.

Proof. Let x_α be a fuzzy point and C be a closed fuzzy set in $\tau_1(\tau_2)$ such that $x_\alpha \in C^c$.

Therefore, C is a closed fuzzy set in $\tau_1(\tau_2)$.

$\Rightarrow 1 - C$ is an open fuzzy set in $\tau_1(\tau_2)$.

$\Rightarrow 1 - C$ is an open fuzzy set in τ_2 (since $\tau_1(\tau_2) \subseteq \tau_2$).

$\Rightarrow C$ is closed in τ_2 .

$\Rightarrow$ open fuzzy sets A_1 and A_2 exist in τ_2 such that $x_\alpha \in A_1$ and $C \subseteq A_2$ and $A_1 \subseteq 1 - A_2$.

Next, for any fuzzy set B such that BqA_1

$\Rightarrow B(z) + A_1(z) > 1$, for some $z \in X$

$\Rightarrow A_1(z) > 1 - B(z)$

$\Rightarrow A_1(z) > r$, where $r = 1 - B(z)$.

The fuzzy regularity of τ_2 guarantees that $\exists A_2 \in \tau_2$ such that $A_2(z) > r$ and $A_2 \subseteq \tau_2 - cl(A_1 \subseteq A_1$, since $\tau_1 \supseteq \tau_2$.

Therefore,

$$\begin{aligned} \tau_1 - cl(A_2) &= \cap \{B \mid A_2 \subseteq B, \text{ where } B \text{ is a closed fuzzy set} \\ &\text{in } \tau_1\} \\ &\subseteq \cap \{B \mid A_2 \subseteq B, B \text{ is a closed fuzzy set in } \tau_2\} \\ &= \tau_2 - cl(A_2). \text{ [Every closed fuzzy set in } \tau_2 \text{ is a} \\ &\text{closed fuzzy set in } \tau_1]. \end{aligned}$$

Therefore, A_2qB and $A_2 \subseteq \tau_1 - cl(A_2) \subseteq A_1$.

Furthermore, $A_1 \in \tau_1(\tau_2)$, and similarly, we can show that $A_2 \in \tau_1(\tau_2)$.

$\Rightarrow$ open fuzzy sets A_1 and A_2 exist in $\tau_1(\tau_2)$ such that $x_\alpha \in A_1$ and $C \subseteq A_2$, and $A_1 \subseteq 1 - A_2$.

Hence, $(X, \tau_1(\tau_2))$ is a fuzzy regular space.

Corollary 1. For any non-empty crisp set X , if $\tau_2 \subseteq \tau_1(\tau_2)$, then $\tau_2 = \tau_1(\tau_2)$ because $\tau_1(\tau_2) \subseteq \tau_2$.

Next, we consider a mixed fuzzy topology comprising fuzzy regular and fuzzy normal.

Example 4. Let $X = \{x, y\}$ and $x \neq y$. Let τ_1 be the fuzzy topology generated by the base $\mathfrak{B}_1 = \{x_\alpha \mid \alpha \in (0, 1]\} \cup \{y_\beta \mid \beta \in (0, 1]\} \cup \{\emptyset\}$ and τ_2 be the fuzzy topology generated by the base $\mathfrak{B}_2 = \{x_\alpha \mid \alpha \in (0, 1], \text{ where } \alpha \text{ is a rational number}\} \cup \{y_\beta \mid \beta \in (0, 1] \text{ and } \beta \text{ is a rational number}\} \cup \{\emptyset\}$. Therefore, τ_1 and τ_2 are fuzzy regular and fuzzy normal spaces, respectively. Next, we show that the mixed fuzzy topology $\tau_1(\tau_2)$ is the same as the fuzzy topology τ_2 . We know that $\tau_1(\tau_2) \subseteq \tau_2$. Moreover, for any element in the base, e.g., $B = \{x_\alpha \mid \alpha \in (0, 1], \text{ where } \alpha \text{ is a rational number}\}$, the fuzzy set $C = \{(x, 1 - \alpha), (y, 1)\}$ is an open fuzzy set in τ_1 and a complement of B in τ_1 . Therefore B is a closed fuzzy set in τ_1 , and τ_1 -closure of $B = B$. Therefore, for any fuzzy set A with $AqB \exists \tau_2$ open fuzzy set B such that AqB and τ_1 -closure of $B = B$. Therefore, $B \in \tau_1(\tau_2)$. Similarly, any base element $\{y_\beta \mid \beta \in (0, 1], \text{ where } \beta \text{ is a rational number}\}$ is also in $\tau_1(\tau_2)$; this implies $\tau_2 \subseteq \tau_1(\tau_2)$. Therefore, $\tau_1(\tau_2) = \tau_2$.

Next, we prove the fuzzy normality and fuzzy regularity of $\tau_1(\tau_2)$.

For the fuzzy regularity, let x be any fuzzy point and B be a closed fuzzy set in $\tau_1(\tau_2)$ such that

$$\begin{aligned} x &\in B^c \\ \Rightarrow B^c(x) &\geq \lambda \\ \Rightarrow 1 - B(x) &\geq \lambda \\ \Rightarrow B(x) &\leq 1 - \lambda \\ \Rightarrow B(x) &= 1 - \lambda \text{ or } B(x) < 1 - \lambda. \end{aligned}$$

If $B(x) = 1 - \lambda$, then $1 - \lambda$ is a rational number.

$B \subseteq \{(x, 1 - \lambda), (y, 1)\} = V$ is an open fuzzy set in $\tau_1(\tau_2)$ and $x \in U = \{(x, \lambda), (y, 0)\} \in \tau_1(\tau_2)$ such that $U = \{(x, \lambda), (y, 0)\} \subseteq 1 - V = \{(x, \lambda), (y, 0)\}$. Therefore, in this case, $\tau_1(\tau_2)$ is fuzzy regular.

Next, if $B(x) < 1 - \lambda$, then $\exists$ is a rational number δ such that $B(x) < \delta < 1 - \lambda$.

Therefore, $B \subseteq \{(x, \delta), (y, 1)\} = V$ is an open fuzzy set in $\tau_1(\tau_2)$.

Furthermore,

$$\begin{aligned} \delta &< 1 - \lambda \\ \Rightarrow 1 - \delta &> 1 - (1 - \lambda) \\ \Rightarrow 1 - \delta &> \lambda \end{aligned}$$

$\exists$ is a rational number r such that $1 - \delta > r > \lambda$

$\Rightarrow \{(x, r), (y, 0)\} = U$ is an open set in $\tau_1(\tau_2)$ such that $x \in U$ and $V^c = \{(x, 1 - \delta), (y, 0)\}$, resulting in $U \subseteq V^c$; therefore, $\tau_1(\tau_2)$ is fuzzy regular in both cases.

Next, we prove that $\tau_1(\tau_2)$ is fuzzy normal.

Let B be any closed fuzzy set and U be an open fuzzy set such that $B \subseteq U$.

$\Rightarrow B(x) \leq U(x), \forall x \in X$ and U is an open fuzzy set in $\tau_1(\tau_2)$. $B(x) = 1 - r$ for some rational number $r \in (0, 1]$

$\Rightarrow B(x) \leq \frac{B(x)+U(x)}{2} \leq U(x)$ and $\frac{B(x)+U(x)}{2}$ is a rational number in $(0, 1]$.

Therefore, $V(x) = \frac{B(x)+U(x)}{2}$ and $V(y) = \frac{B(y)+U(y)}{2}$ defines a fuzzy open set in $\tau_1(\tau_2)$.

Furthermore, $B(x) \leq V(x) \leq U(x) \forall x \in X$. Similarly, $1 - V$ yields rational values for both x and y ; therefore, $1 - V$ is open in $\tau_1(\tau_2) \Rightarrow V$ is fuzzy closed as well as $\Rightarrow V(x) = \overline{V}(x) \forall x \in X$.

$$\begin{aligned} \Rightarrow B(x) &\leq V(x) \leq \overline{V}(x) \leq U(x) \forall x \in X \\ \Rightarrow B(x) &\leq V(x) \leq \overline{V}(x) \leq U(x). \end{aligned}$$

Therefore, $\tau_1(\tau_2)$ is a normal mixed fuzzy topological space.

Theorem 10. If $\tau_1 \supseteq \tau_2$ and τ_2 are fuzzy normal, then $\tau_1(\tau_2)$ is also fuzzy normal.

Proof. Since τ_2 is fuzzy normal, it is fuzzy regular. Therefore, from Corollary 1, we obtain $\tau_1(\tau_2) = \tau_2$. Hence, $\tau_1(\tau_2)$ is also normal. $\square$

Theorem 11. A fuzzy T_2 mixed fts $(X, \tau_1(\tau_2))$ is a mixed fuzzy T_1 -space, and a mixed fuzzy T_1 space is a mixed fuzzy T_0 -space.

Proof. Let $(X, \tau_1(\tau_2))$ be a T_2 mixed fuzzy topological space.

Let x_α and y_β be two fuzzy points such that $x \neq y$. Therefore, open fuzzy sets O_1 and O_2 exist such that $x_\alpha \in O_1 \subseteq co(y_\beta)$, $y_\beta \in O_2 \subseteq co(x_\alpha)$, and $O_1 \subseteq co(O_2)$. Therefore, $x_\alpha \in O_1 \subseteq co(y_\beta)$ and $y_\beta \in O_2 \subseteq co(x_\alpha)$; as such, $(X, \tau_1(\tau_2))$ is a T_1 mixed fuzzy topological space.

Let $(X, \tau_1(\tau_2))$ be a T_1 mixed fuzzy topological space.

Let x_α and y_β be two fuzzy points such that $x \neq y$. Therefore, open fuzzy sets O_1 and O_2 exist such that $x_\alpha \in O_1 \subseteq co(y_\beta)$ and $y_\beta \in O_2 \subseteq co(x_\alpha)$. Consequently, $x_\alpha \in O_1 \subseteq co(y_\beta)$ or $y_\beta \in O_2 \subseteq co(x_\alpha)$. Hence, $(X, \tau_1(\tau_2))$ is a mixed fuzzy T_0 topological space. $\square$

4. Conclusion

In this study, we defined fuzzy- T_0 , fuzzy- T_1 , fuzzy- T_2 (or Hausdorff), fuzzy regular, and fuzzy normal spaces in mixed fuzzy topological spaces. We established the relationships among fuzzy- T_0 , fuzzy- T_1 , fuzzy- T_2 , fuzzy regular, and fuzzy normal spaces in mixed fuzzy topological spaces. Only a few theorems have been established to investigate separation axioms in mixed fuzzy topological spaces. More properties and theorems pertaining to separation axioms will be established in the near future using these established results in the context of mixed fuzzy topological spaces.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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