On Picture Fuzzy Ideals of Near-Rings

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Abstract

In managing vagueness and complexity, the theory of picture fuzzy sets has important advantages. In addition, picture fuzzy data are useful for modeling the uncertain existence of subjective decisions and for more flexibly assessing the fuzziness and imprecision. The goal of this study is to establish and discuss their desirable properties by defining the picture fuzzy ideals of a near-ring. Based on new definitions, this paper presents some examples and properties of picture fuzzy subsets of such rings.

Keywords: Near-ring, Picture fuzzy set, Picture fuzzy ideals, Homomorphism of the near-rings

1. Introduction

The definition of a fuzzy set with a membership function μ was introduced by Zadeh [1]. This assigns a number from the unit interval $[0,1]$ to each member of the universe of discourse to indicate the degree of belonging to the collection under consideration. The principle of fuzzy sets generalizes the theory of classical sets by facilitating intermediate situations between the whole and nothing. A membership function in a fuzzy set is introduced to characterize the membership degree of an entity in a class. The degree of membership lies between values of 0 and 1, which indicate that the element does not belong and belongs to a class, respectively. The remaining values then indicate the degree to which the class belongs. In fuzzy set theory, the membership function has replaced the characteristic function of the crisp set theory. Since Zadeh's groundbreaking study [1], the theory of fuzzy sets has been used in various fields such as artificial intelligence, automata theory, engineering, life sciences, mathematics, management sciences, medical sciences, signal processing, social sciences, and statistics [2].

The definition of the theory of fuzzy sets seems to be inconclusive because of the absence of a non-membership function and disregard for the margin of hesitation possibility. These shortcomings were critically analyzed by Atanassov's intuitionistic fuzzy set (IFS) [3] including one more component called the non-membership function into the fuzzy set. The author's theory has gained extensive recognition as an extremely valuable tool in the fields of science, technology, engineering, and medicine. One of the extensions of IFS is the picture fuzzy set (PFS), which was developed by Cuong [4].

In PFS theory, there are three components: positive, abstinence (neutral), and negative terms, and the sum of the three components is less than or equal to 1. Simultaneously, the PFS has

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been applied in the fields of science and engineering. The development of the PFS can be found in many fields [5–12].

In real life, there are more than one opinion for every decision. For instance, in a democratic voting system, 10,000 individuals might vote in an election. If the election commission distributes 10,000 paper ballots, each voter can only use one ballot to cast his or her vote. The election results are generally divided into four categories based on the number of paper ballots cast: “vote for the candidate (5,000),” “vote against the candidate (2,000),” “abstain from voting (2,000),” and “refuse to vote (1,000).” “Abstain from voting” indicates that the paper ballot remains blank, which contradicts both “vote for the candidate” and “vote against candidate” but does consider the possibility of voting. However, “refuse to vote” means bypassing the vote altogether. Neutrality is related to “vote for the candidate” and “vote against candidate.” A refusal is related to bypassing the vote (that is, “refuse to vote”). PFS deals with these types of cases.

The advantages and limitations of PFS are as follows:
Advantages:

- The human opinions including yes, abstain, no, and refusal can be expressed.
- The PFS can be transformed into IFS and FS.

Limitations:

- It cannot express the evaluation information when decision makers have difficulty determining an accurate value of each membership level.

Abou-zaid [13] studied the concept of fuzzy sub-near-rings and fuzzy ideals of near-rings. Davvaz [14] proposed the notion of fuzzy ideals of near-rings with interval-value membership functions in 2001. Several studies [15–17] further explored this notion.

The notion of anti-fuzzy subgroups of groups was introduced by Biswas in [18], and the concept of anti-fuzzy R-subgroups of the near-ring was studied by Jun et al. [19]. In addition, Kim and Jun [20] introduced the notion of anti-fuzzy ideals in near-rings in an intuitionistic fuzzy environment and studied the notion of generalized anti-fuzzy bi-ideals in ordered semigroups. Several researchers have also studied the ring concept in terms of intuitionistic fuzzy environments [21,22].

The concept of fuzzy group theory was the first to be applied to many algebraic structures, and was introduced by Rosenfeld [23]. Later, the fuzzy ideals of rings were introduced

and studied by Lui [24], and many researchers [25–29] have extended these concepts to the extensions of fuzzy sets. The concepts of near-rings and their properties are interesting in the field of algebraic structures. The concept of PFS theory specializes in dealing with the uncertainty parameter (neutrality), which is not found in fuzzy or IFS theories. Asif et al. [30] introduced and studied the basic concept of the picture fuzzy ideals of near-rings. This motivated us to introduce and investigate more properties of the picture fuzzy ideals of near-rings.

This study is concerned about picture fuzzy ideals of near-rings and is constructed as follows: in Section 2, we present some definitions that are used throughout the paper. In Section 3, we introduce and study the notion of picture fuzzy ideals of a near-ring and discuss the essential characteristics of near-ring picture fuzzy ideals. Moreover, we studied picture fuzzy ideals under different operations and near-ring homomorphisms.

2. Preliminaries

We include some descriptions, comments, and findings in this section that are important and are regularly used throughout the paper.

A description of the picture fuzzy structure was introduced by Cuong [4] as follows:

Definition 1 [18]. Let $\mathcal{U}$ be a non-empty set. Then, a picture fuzzy set (PFS) $\mathfrak{S}$ on $\mathcal{U}$ is defined as

$$\mathfrak{S} = \{(\zeta, \langle m(\zeta), a(\zeta), n(\zeta) \rangle) : \zeta \in \mathcal{U}\},$$

where $m, a, n : \mathcal{U} \rightarrow [0, 1]$ are called positive, abstinence, and negative membership functions, respectively. As a condition for a PFS, $0 \leq m(\zeta) + a(\zeta) + n(\zeta) \leq 1$, and the degree of refusal is defined as $r(\zeta) = 1 - (m(\zeta) + a(\zeta) + n(\zeta))$. A picture fuzzy number is defined as a triplet (m, a, n) .

Definition 2 [8]. Let $\mathfrak{S}_1 = \{(\zeta, \langle m_{\mathfrak{S}_1}(\zeta), a_{\mathfrak{S}_1}(\zeta), n_{\mathfrak{S}_1}(\zeta) \rangle) : \zeta \in \mathcal{U}\}$ and $\mathfrak{S}_2 = \{(\zeta, \langle m_{\mathfrak{S}_2}(\zeta), a_{\mathfrak{S}_2}(\zeta), n_{\mathfrak{S}_2}(\zeta) \rangle) : \zeta \in \mathcal{U}\}$ be two PFSs described on $\mathcal{U}$, which is the universe of discourse. Then,

1. the union of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ is

$$\mathfrak{S}_1 \cup \mathfrak{S}_2 = \{(\zeta, \langle m_{\mathfrak{S}_1}(\zeta) \vee m_{\mathfrak{S}_2}(\zeta), a_{\mathfrak{S}_1}(\zeta) \wedge a_{\mathfrak{S}_2}(\zeta), n_{\mathfrak{S}_1}(\zeta) \wedge n_{\mathfrak{S}_2}(\zeta) \rangle) : \zeta \in \mathcal{U}\},$$

2. the intersection of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ is

$$\mathfrak{S}_1 \cap \mathfrak{S}_2 = \{(\zeta, \langle m_{\mathfrak{S}_1}(\zeta) \wedge m_{\mathfrak{S}_2}(\zeta), a_{\mathfrak{S}_1}(\zeta) \vee a_{\mathfrak{S}_2}(\zeta), n_{\mathfrak{S}_1}(\zeta) \vee n_{\mathfrak{S}_2}(\zeta) \rangle) : \zeta \in \mathcal{U}\}.$$

$$n_{\mathfrak{S}_1}(\zeta) \vee n_{\mathfrak{S}_2}(\zeta)) : \zeta \in \mathfrak{U}\},$$

3. the symmetric difference of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ is

$$\mathfrak{S}_1 - \mathfrak{S}_2 = \{(\zeta, \langle m_{\mathfrak{S}_1 - \mathfrak{S}_2}(\zeta), a_{\mathfrak{S}_1 - \mathfrak{S}_2}(\zeta), n_{\mathfrak{S}_1 - \mathfrak{S}_2}(\zeta) \rangle) : \zeta \in \mathfrak{U}\},$$

where

$$\begin{aligned} m_{\mathfrak{S}_1 - \mathfrak{S}_2}(\zeta) &= 0 \vee m_{\mathfrak{S}_1} - m_{\mathfrak{S}_2}, \\ n_{\mathfrak{S}_1 - \mathfrak{S}_2}(\zeta) &= 0 \vee n_{\mathfrak{S}_1} - n_{\mathfrak{S}_2}, \\ a_{\mathfrak{S}_1 - \mathfrak{S}_2}(\zeta) &= \begin{cases} 1 - m_{\mathfrak{S}_1 - \mathfrak{S}_2}(\zeta) - n_{\mathfrak{S}_1 - \mathfrak{S}_2}(\zeta), \\ \text{if } a_{\mathfrak{S}_1}(\zeta) > a_{\mathfrak{S}_2}(\zeta), \\ \{1 + a_{\mathfrak{S}_1}(\zeta) - a_{\mathfrak{S}_2}(\zeta) \\ \wedge \{1 - m_{\mathfrak{S}_1 - \mathfrak{S}_2}(\zeta) - n_{\mathfrak{S}_1 - \mathfrak{S}_2}(\zeta)\}, \\ \text{if } a_{\mathfrak{S}_1}(\zeta) \leq a_{\mathfrak{S}_2}(\zeta), \end{cases} \end{aligned}$$

4. $\mathfrak{S}_1 \subseteq \mathfrak{S}_2$ if and only if

$$m_{\mathfrak{S}_1}(\zeta) \leq m_{\mathfrak{S}_2}(\zeta), \quad a_{\mathfrak{S}_1}(\zeta) \geq a_{\mathfrak{S}_2}(\zeta),$$

and

$$n_{\mathfrak{S}_1}(\zeta) \geq n_{\mathfrak{S}_2}(\zeta), \quad \forall \zeta \in \mathfrak{U}.$$

Definition 3. The PFS-set is not a set, but is a particular unit of certain components of the set $PF(\hat{\mathfrak{U}})$ and can therefore be positioned as a set of ordered pairs: $(\hat{\mathfrak{F}}, \hat{\mathfrak{E}}) = \{(\zeta, \langle \zeta, m_{\hat{\mathfrak{F}}(\xi)}(\zeta), a_{\hat{\mathfrak{F}}(\xi)}(\zeta), n_{\hat{\mathfrak{F}}(\xi)}(\zeta) \rangle \mid \zeta \in \mathfrak{U} : \xi \in \hat{\mathfrak{E}}\}$, where $m_{\hat{\mathfrak{F}}(\xi)}(\zeta)$, $a_{\hat{\mathfrak{F}}(\xi)}(\zeta)$, and $n_{\hat{\mathfrak{F}}(\xi)}(\zeta)$ are the positive, abstinence, and negative membership functions, respectively. If $\zeta \in \hat{\mathfrak{U}}$, $m_{\hat{\mathfrak{F}}(\xi)}(\zeta) + a_{\hat{\mathfrak{F}}(\xi)}(\zeta) + n_{\hat{\mathfrak{F}}(\xi)}(\zeta) \leq 1$.

Definition 4 [31]. A near-ring N is a non-void set, with ‘+’ and ‘.’ as two binary operations with the following constraints:

- (i) $(N, +)$ is a group;
- (ii) $(N, \cdot)$ is a semi-group; and
- (iii) $u \cdot (v + w) = u \cdot v + u \cdot w$, for all $u, v, w \in N$.

To be precise, because N fulfills the left distributive law, it is a left near-ring. Instead of a “left near-ring,” the term we use is a “near-ring.” Recall that $u \cdot 0 = 0$ and $u \cdot (-v) = -uv$ for all $u, v \in N$; however, usually $0 \cdot u \neq 0$ for some $u \in N$.

Definition 5 [31]. A non-empty set $\mathbb{I}$ of a near-ring N is an ideal of N if the following axioms are satisfied:

- (i) $(\mathbb{I}, +)$ is a normal subgroup of $(N, +)$;

- (ii) $N\mathbb{I} \subseteq \mathbb{I}$; and

- (iii) $(u + a)v - uv \in \mathbb{I}$ for all $a \in \mathbb{I}$ and all $u, v \in N$.

If $\mathbb{I}$ satisfies (i) and (ii) of Definition 5, then $\mathbb{I}$ is a left ideal of N . If $\mathbb{I}$ satisfies (i) and (iii) of Definition 5, then $\mathbb{I}$ is the right ideal of N .

Definition 6 [13]. A fuzzy set $\mathfrak{A} \in N$, where N is a near-ring, is said to be a fuzzy subnear-ring of N if

- (i) $\mu(u - v) \geq \mu(u) \wedge \mu(v)$,
- (ii) $\mu(uv) \geq \mu(u) \wedge \mu(v)$, for all $u, v \in N$

Definition 7 [32]. A fuzzy set $\mathfrak{A} \in N$, where N is a near-ring, is called the fuzzy ideal of N , if the following axioms are satisfied:

- (i) $\mu(u - v) \geq \min\{\mu(u), \mu(v)\}$;
- (ii) $\mu(v + u - v) \geq \mu(u)$;
- (iii) $\mu(uv) \geq \mu(v)$;
- (iv) $\mu((u + w)v - uv) \geq \mu(w)$ for all $u, v, w \in N$.

Let $\mathfrak{A}$ be a fuzzy set. If it satisfies (i)–(iii), then it is said to be a fuzzy left ideal of N , and if it satisfies (i), (ii) and (iv), then it is said to be a fuzzy right ideal of N .

Definition 8. A fuzzy set $\mathfrak{A} \in N$, where N is a near-ring, is said to be an anti-fuzzy ideal of N , if the following axioms are satisfied:

- (i) $\mu(u - v) \leq \mu(u) \wedge \mu(v)$;
- (ii) $\mu(v + u - v) \leq \mu(u)$;
- (iii) $\mu(uv) \leq \mu(v)$;
- (iv) $\mu((u + w)v - uv) \leq \mu(w)$ for all $u, v, w \in N$.

3. Picture Fuzzy Subsets of Near-Rings

In this section, we present new concepts related to the picture fuzzy subnear-rings. In particular, we define and study the picture fuzzy ideals of the near-rings.

3.1 Picture Fuzzy Ideals of Near-Rings: Definitions and Examples

Definition 9. A picture fuzzy set $\mathfrak{P}$ in a near-ring N is said to be a picture fuzzy subnear-ring of N if

- (i) $m_{\mathfrak{P}}(u - v) \geq m_{\mathfrak{P}}(u) \wedge m_{\mathfrak{P}}(v)$, $a_{\mathfrak{P}}(u - v) \leq a_{\mathfrak{P}}(u) \vee a_{\mathfrak{P}}(v)$, $n_{\mathfrak{P}}(u - v) \leq n_{\mathfrak{P}}(u) \vee n_{\mathfrak{P}}(v)$; and
- (ii) $m_{\mathfrak{P}}(uv) \geq m_{\mathfrak{P}}(u) \wedge m_{\mathfrak{P}}(v)$, $a_{\mathfrak{P}}(uv) \leq a_{\mathfrak{P}}(u) \vee a_{\mathfrak{P}}(v)$, $n_{\mathfrak{P}}(uv) \leq n_{\mathfrak{P}}(u) \vee n_{\mathfrak{P}}(v)$.

Definition 10. Let N be a nearby ring. A picture fuzzy set $\mathfrak{P}$ in a near-ring N is said to be a picture fuzzy ideal of N if the following axioms are satisfied: For all $u, v, w \in N$,

- (i) $m_{\mathfrak{P}}(u - v) \geq m_{\mathfrak{P}}(u) \wedge m_{\mathfrak{P}}(v)$, $a_{\mathfrak{P}}(u - v) \leq a_{\mathfrak{P}}(u) \vee a_{\mathfrak{P}}(v)$, $n_{\mathfrak{P}}(u - v) \leq n_{\mathfrak{P}}(u) \vee n_{\mathfrak{P}}(v)$;
- (ii) $m_{\mathfrak{P}}(v + u - v) \geq m_{\mathfrak{P}}(u)$, $a_{\mathfrak{P}}(v + u - v) \leq a_{\mathfrak{P}}(u)$, $n_{\mathfrak{P}}(v + u - v) \leq n_{\mathfrak{P}}(u)$;
- (iii) $m_{\mathfrak{P}}(uv) \geq m_{\mathfrak{P}}(v)$, $a_{\mathfrak{P}}(uv) \leq a_{\mathfrak{P}}(v)$, $n_{\mathfrak{P}}(uv) \leq n_{\mathfrak{P}}(v)$; and
- (iv) $m_{\mathfrak{P}}((u + w)v - uv) \geq m_{\mathfrak{P}}(w)$, $a_{\mathfrak{P}}((u + w)v - uv) \leq a_{\mathfrak{P}}(w)$, $n_{\mathfrak{P}}((u + w)v - uv) \leq n_{\mathfrak{P}}(w)$.

A picture fuzzy subset is said to be a picture fuzzy left ideal of N if it satisfies (i)–(iii), and a picture fuzzy subset is said to be a picture fuzzy right ideal of N if it satisfies (i), (ii), and (iv).

Remark 1. Let $(N, +, \cdot)$ be any near-ring, and $\mathfrak{P}$ be a PFS on N . If $\mathfrak{P}$ is a picture fuzzy ideal of N , it is then a picture fuzzy left ideal of N (a picture fuzzy right ideal of N), and as a result, it is a picture fuzzy subnear-ring of N .

Proposition 1. Let N be a near-ring, and $\mathfrak{P}$ be a PFS of N . If $\mathfrak{P}$ is a picture fuzzy subnear-ring (left ideal or right ideal or ideal) of N , then the following statements are true:

- (1) $m_{\mathfrak{P}}(0) \geq m_{\mathfrak{P}}(u)$, $a_{\mathfrak{P}}(0) \leq a_{\mathfrak{P}}(u)$, $n_{\mathfrak{P}}(0) \leq n_{\mathfrak{P}}(u)$ for all $u \in N$; and
- (2) $m_{\mathfrak{P}}(-u) = m_{\mathfrak{P}}(u)$, $a_{\mathfrak{P}}(-u) = a_{\mathfrak{P}}(u)$, $n_{\mathfrak{P}}(-u) = n_{\mathfrak{P}}(u)$ for all $u \in N$.

Proof. (1) Let $\mathfrak{P}$ be a picture fuzzy subnear-ring (left ideal, right ideal, or ideal) of N . Then, condition (i) of Definition 9 holds. This implies that $m_{\mathfrak{P}}(0) = m_{\mathfrak{P}}(u - u) \geq m_{\mathfrak{P}}(u)$, $a_{\mathfrak{P}}(0) = a_{\mathfrak{P}}(u - u) \leq a_{\mathfrak{P}}(u)$, and $n_{\mathfrak{P}}(0) = n_{\mathfrak{P}}(u - u) \leq n_{\mathfrak{P}}(u)$.

(2) Substituting $u = 0$ in Condition (i) of Definition 9, we obtain $m_{\mathfrak{P}}(-v) \geq m_{\mathfrak{P}}(v)$, $a_{\mathfrak{P}}(-v) \leq a_{\mathfrak{P}}(v)$, $n_{\mathfrak{P}}(-v) \leq n_{\mathfrak{P}}(v)$ for all $v \in N$. Because $-(-v) = v$, it follows that $m_{\mathfrak{P}}(-v) \leq m_{\mathfrak{P}}(v)$, $a_{\mathfrak{P}}(-v) \geq a_{\mathfrak{P}}(v)$, and $n_{\mathfrak{P}}(-v) \geq n_{\mathfrak{P}}(v)$ for all $v \in N$. This completes the proof. $\square$

Remark 2. Let $(N, +, \cdot)$ be any near-ring, a, b, c be fixed values in the unit interval with $0 \leq a + b + c \leq 1$, and define the PFS $\mathfrak{P}$ on N as follows:

$$m_{\mathfrak{P}}(u) = a, a_{\mathfrak{P}}(u) = b, n_{\mathfrak{P}}(u) = c \text{ for all } u \in N.$$

Then, $\mathfrak{P}$ is a picture fuzzy ideal of N .

Example 1. Let $N_1 = \{0, 1, 2, 3\}$ with the two binary operations “ $+_1$ ” and “ $\cdot_1$ ” be defined as follows:

$+_1$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

and

$\cdot_1$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	1	2	3
3	0	1	2	3

Then, $(N_1, +_1, \cdot_1)$ is a near-ring. Let the picture fuzzy set $\mathfrak{P}_1 = (m_{\mathfrak{P}_1}, a_{\mathfrak{P}_1}, n_{\mathfrak{P}_1})$ be defined as $m_{\mathfrak{P}_1}(0) = 0.8$, $m_{\mathfrak{P}_1}(1) = m_{\mathfrak{P}_1}(2) = m_{\mathfrak{P}_1}(3) = 0.4$, $a_{\mathfrak{P}_1}(0) = 0.15$, $a_{\mathfrak{P}_1}(1) = a_{\mathfrak{P}_1}(2) = a_{\mathfrak{P}_1}(3) = 0.2$, $n_{\mathfrak{P}_1}(0) = 0.02$, $n_{\mathfrak{P}_1}(1) = n_{\mathfrak{P}_1}(2) = n_{\mathfrak{P}_1}(3) = 0.3$. It can be clearly seen that the picture fuzzy set $\mathfrak{P}_1 = (m_{\mathfrak{P}_1}, a_{\mathfrak{P}_1}, n_{\mathfrak{P}_1})$ is a picture fuzzy ideal of N_1 .

Example 2. Let $\mathbb{Z}$ be the set of integers. Then, $(\mathbb{Z}, +, \cdot)$ is a near-ring. Here, “ $+$ ” is a standard addition and “ $\cdot$ ” is defined as $a \cdot b = b$ for all $a, b \in \mathbb{Z}$. We define the picture fuzzy set $\mathfrak{P} = (m_{\mathfrak{P}}, a_{\mathfrak{P}}, n_{\mathfrak{P}})$ on $\mathbb{Z}$ as follows: For all $x \in \mathbb{Z}$,

$$m_{\mathfrak{P}}(x) = \begin{cases} 0.8 & \text{if } x \text{ is a multiple of 4;} \\ 0.2 & \text{otherwise,} \end{cases}$$

$$a_{\mathfrak{P}}(x) = \begin{cases} 0.1 & \text{if } x \text{ is a multiple of 4;} \\ 0.35 & \text{otherwise,} \end{cases}$$

and

$$n_{\mathfrak{P}}(x) = \begin{cases} 0.05 & \text{if } x \text{ is a multiple of 4;} \\ 0.4 & \text{otherwise,} \end{cases}$$

PFS $\mathfrak{P}' = (m_{\mathfrak{P}'}, a_{\mathfrak{P}'}, n_{\mathfrak{P}'})$ on $\mathbb{Z}$ is as follows: For all $x \in \mathbb{Z}$,

$$m_{\mathfrak{P}'}(x) = \begin{cases} 0.7 & \text{if } x \text{ is a multiple of 3;} \\ 0.3 & \text{otherwise,} \end{cases}$$

$$a_{\mathfrak{P}'}(x) = \begin{cases} 0.05 & \text{if } x \text{ is a multiple of 3;} \\ 0.36 & \text{otherwise,} \end{cases}$$

and

$$n_{\mathfrak{P}'}(x) = \begin{cases} 0.1 & \text{if } x \text{ is a multiple of 3;} \\ 0.33 & \text{otherwise.} \end{cases}$$

It can be clearly seen that the picture fuzzy sets $\mathfrak{P}, \mathfrak{P}'$ are picture fuzzy ideals of $\mathbb{Z}$.

Example 3. Let $N_2 = \{e, f, g, h\}$ with the two binary operations “ $+_2$ ” and “ $\cdot_2$ ” be defined as follows:

$+_2$	e	f	g	h
e	e	f	g	h
f	f	e	h	g
g	g	f	h	e
h	h	g	e	f

and

$\cdot_2$	e	f	g	h
e	e	e	e	e
f	e	e	e	e
g	e	e	e	e
h	e	e	f	f

Then, $(N_2, +_2, \cdot_2)$ is a near-ring. Let the picture fuzzy set $\mathfrak{P}_2 = (m_{\mathfrak{P}_2}, a_{\mathfrak{P}_2}, n_{\mathfrak{P}_2})$ be defined as follows: $m_{\mathfrak{P}_2}(e) = 0.7$, $a_{\mathfrak{P}_2}(e) = 0.15$, $n_{\mathfrak{P}_2}(e) = 0.15$, $m_{\mathfrak{P}_2}(f) = 0.5$, $a_{\mathfrak{P}_2}(f) = 0.2$, $n_{\mathfrak{P}_2}(f) = 0.2$, $m_{\mathfrak{P}_2}(g) = m_{\mathfrak{P}_2}(h) = 0.2$, $a_{\mathfrak{P}_2}(g) = a_{\mathfrak{P}_2}(h) = 0.4$, $n_{\mathfrak{P}_2}(g) = n_{\mathfrak{P}_2}(h) = 0.4$. It can be clearly seen that the picture fuzzy set $\mathfrak{P}_2 = (m_{\mathfrak{P}_2}, a_{\mathfrak{P}_2}, n_{\mathfrak{P}_2})$ is a picture fuzzy ideal of N_2 .

3.2 Operations on Picture Fuzzy Ideals of Near-Rings

In this subsection, we study the different types of operations on picture fuzzy ideals under near-rings.

Theorem 1. If $\mathfrak{A}$ and $\mathfrak{B}$ are any two picture fuzzy ideals of N , then $\mathfrak{A} \cap \mathfrak{B}$ is a picture fuzzy ideal of N .

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two picture fuzzy ideals of N . Let $u, v, w \in N$.

$$(i) \quad m_{\mathfrak{A} \cap \mathfrak{B}}(u - v) = m_{\mathfrak{A}}(u - v) \wedge m_{\mathfrak{B}}(u - v) \geq \{ \{ m_{\mathfrak{A}}(u) \wedge m_{\mathfrak{A}}(v) \} \wedge \{ m_{\mathfrak{B}}(u) \wedge m_{\mathfrak{B}}(v) \} = \{ \{ m_{\mathfrak{A}}(u) \wedge m_{\mathfrak{B}}(v) \} \wedge \{ m_{\mathfrak{A}}(u) \vee m_{\mathfrak{B}}(v) \} \} = \{ m_{\mathfrak{A} \cap \mathfrak{B}}(u) \wedge m_{\mathfrak{A} \cap \mathfrak{B}}(v) \}.$$

$$a_{\mathfrak{A} \cap \mathfrak{B}}(u - v) = a_{\mathfrak{A}}(u - v) \wedge a_{\mathfrak{B}}(u - v) \leq \{ \{ a_{\mathfrak{A}}(u) \wedge a_{\mathfrak{A}}(v) \} \wedge \{ a_{\mathfrak{B}}(u) \wedge a_{\mathfrak{B}}(v) \} = \{ \{ a_{\mathfrak{A}}(u) \wedge a_{\mathfrak{B}}(v) \} \wedge \{ a_{\mathfrak{A}}(u) \vee a_{\mathfrak{B}}(v) \} \} = \{ a_{\mathfrak{A} \cap \mathfrak{B}}(u) \wedge a_{\mathfrak{A} \cap \mathfrak{B}}(v) \}.$$

$$n_{\mathfrak{A} \cap \mathfrak{B}}(u - v) = n_{\mathfrak{A}}(u - v) \vee n_{\mathfrak{B}}(u - v) \leq \{ \{ n_{\mathfrak{A}}(u) \vee n_{\mathfrak{A}}(v) \} \vee \{ n_{\mathfrak{B}}(u) \vee n_{\mathfrak{B}}(v) \} = \{ \{ n_{\mathfrak{A}}(u) \vee n_{\mathfrak{B}}(v) \} \vee \{ n_{\mathfrak{A}}(u) \vee n_{\mathfrak{B}}(v) \} \} = \{ n_{\mathfrak{A} \cap \mathfrak{B}}(u) \vee n_{\mathfrak{A} \cap \mathfrak{B}}(v) \}.$$

$$(ii) \quad m_{\mathfrak{A} \cap \mathfrak{B}}(v + u - v) = m_{\mathfrak{A}}(v + u - v) \wedge m_{\mathfrak{B}}(v + u - v) \geq \{ m_{\mathfrak{A}}(u) \wedge m_{\mathfrak{B}}(u) \} = m_{\mathfrak{A} \cap \mathfrak{B}}(u).$$

$$a_{\mathfrak{A} \cap \mathfrak{B}}(v + u - v) = a_{\mathfrak{A}}(v + u - v) \wedge a_{\mathfrak{B}}(v + u - v) \leq \{ a_{\mathfrak{A}}(u) \wedge a_{\mathfrak{B}}(u) \} = a_{\mathfrak{A} \cap \mathfrak{B}}(u).$$

$$n_{\mathfrak{A} \cap \mathfrak{B}}(v + u - v) = n_{\mathfrak{A}}(v + u - v) \vee n_{\mathfrak{B}}(v + u - v) \leq \{ n_{\mathfrak{A}}(u) \vee n_{\mathfrak{B}}(u) \} = n_{\mathfrak{A} \cap \mathfrak{B}}(u).$$

$$(iii) \quad m_{\mathfrak{A} \cap \mathfrak{B}}(uv) = m_{\mathfrak{A}}(uv) \wedge m_{\mathfrak{B}}(uv) \geq \{ m_{\mathfrak{A}}(v) \wedge m_{\mathfrak{B}}(v) \} = m_{\mathfrak{A} \cap \mathfrak{B}}(v).$$

$$a_{\mathfrak{A} \cap \mathfrak{B}}(uv) = a_{\mathfrak{A}}(uv) \wedge a_{\mathfrak{B}}(uv) \leq \{ a_{\mathfrak{A}}(v) \wedge a_{\mathfrak{B}}(v) \} = a_{\mathfrak{A} \cap \mathfrak{B}}(v).$$

$$n_{\mathfrak{A} \cap \mathfrak{B}}(uv) = n_{\mathfrak{A}}(uv) \vee n_{\mathfrak{B}}(uv) \leq \{ n_{\mathfrak{A}}(v) \vee n_{\mathfrak{B}}(v) \} = n_{\mathfrak{A} \cap \mathfrak{B}}(v).$$

$$(iv) \quad m_{\mathfrak{A} \cap \mathfrak{B}}((u + w)v - uv) = m_{\mathfrak{A}}((u + w)v - uv) \wedge m_{\mathfrak{B}}((u + w)v - uv) \geq \{ m_{\mathfrak{A}}(w) \wedge m_{\mathfrak{B}}(w) \} = m_{\mathfrak{A} \cap \mathfrak{B}}(w).$$

$$a_{\mathfrak{A} \cap \mathfrak{B}}((u + w)v - uv) = a_{\mathfrak{A}}((u + w)v - uv) \wedge a_{\mathfrak{B}}((u + w)v - uv) \leq \{ a_{\mathfrak{A}}(w) \wedge a_{\mathfrak{B}}(w) \} = a_{\mathfrak{A} \cap \mathfrak{B}}(w).$$

$$n_{\mathfrak{A} \cap \mathfrak{B}}((u + w)v - uv) = n_{\mathfrak{A}}((u + w)v - uv) \vee n_{\mathfrak{B}}((u + w)v - uv) \leq \{ n_{\mathfrak{A}}(w) \vee n_{\mathfrak{B}}(w) \} = n_{\mathfrak{A} \cap \mathfrak{B}}(w).$$

Therefore, $\mathfrak{A} \cap \mathfrak{B}$ is a picture fuzzy ideal of N . $\square$

Corollary 1. If $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \dots, \mathfrak{A}_n$ are picture fuzzy ideals of N , then $\mathfrak{A} = \bigcap_{i=1}^n \mathfrak{A}_i$ is again a picture fuzzy ideal of N .

Proof. The proof follows from Theorem 1 and applies induction. $\square$

Example 4. Consider $(\mathbb{Z}, +, \cdot)$ as a near-ring, $\mathfrak{P} = (m_{\mathfrak{P}}, a_{\mathfrak{P}}, n_{\mathfrak{P}})$, and let $\mathfrak{P}' = (m_{\mathfrak{P}'}, a_{\mathfrak{P}'}, n_{\mathfrak{P}'})$ be picture fuzzy sets and their corresponding picture fuzzy ideals presented in Example 2. Then, the picture fuzzy set $\mathfrak{P} \cap \mathfrak{P}' = (m_{\mathfrak{P} \cap \mathfrak{P}'}, a_{\mathfrak{P} \cap \mathfrak{P}'}, n_{\mathfrak{P} \cap \mathfrak{P}'})$

on $\mathbb{Z}$ is defined as follows: For all $x \in \mathbb{Z}$,

$$m_{\mathfrak{P} \cap \mathfrak{P}'}(x) = \begin{cases} 0.7, & \text{if } x \text{ is a multiple of 12;} \\ 0.3, & \text{if } x \text{ is a multiple of 4 but is not} \\ & \text{a multiple of 3;} \\ 0.2, & \text{otherwise,} \end{cases}$$

$$a_{\mathfrak{P} \cap \mathfrak{P}'}(x) = \begin{cases} 0.1, & \text{if } x \text{ is a multiple of 12;} \\ 0.35, & \text{if } x \text{ is a multiple of 3 but is not} \\ & \text{a multiple of 4;} \\ 0.36, & \text{otherwise,} \end{cases}$$

and

$$n_{\mathfrak{P} \cap \mathfrak{P}'}(x) = \begin{cases} 0.1, & \text{if } x \text{ is a multiple of 12;} \\ 0.33, & \text{if } x \text{ is a multiple of 4 but is not} \\ & \text{a multiple of 3;} \\ 0.4, & \text{otherwise.} \end{cases}$$

Using Theorem 1, we find that the picture fuzzy set $\mathfrak{P} \cap \mathfrak{P}'$ is a picture fuzzy ideal of $\mathbb{Z}$.

Remark 3. The union of the picture fuzzy ideals of a near-ring N is not necessarily a picture fuzzy ideal of N .

We illustrate Remark 3 through the following example.

Example 5. Consider $(\mathbb{Z}, +, \cdot)$ as a near-ring, $\mathfrak{P} = (m_{\mathfrak{P}}, a_{\mathfrak{P}}, n_{\mathfrak{P}})$, and $\mathfrak{P}' = (m_{\mathfrak{P}'}, a_{\mathfrak{P}'}, n_{\mathfrak{P}'})$ be picture fuzzy sets and their corresponding picture fuzzy ideals presented in Example 2. Then, the picture fuzzy set $\mathfrak{P} \cup \mathfrak{P}' = (m_{\mathfrak{P} \cup \mathfrak{P}'}, a_{\mathfrak{P} \cup \mathfrak{P}'}, n_{\mathfrak{P} \cup \mathfrak{P}'})$ on $\mathbb{Z}$ is not a picture fuzzy ideal of $\mathbb{Z}$. This can be easily observed as $0.3 = m_{\mathfrak{P} \cup \mathfrak{P}'}(1) = m_{\mathfrak{P} \cup \mathfrak{P}'}(4 - 3) \not\geq 0.7 = m_{\mathfrak{P} \cup \mathfrak{P}'}(4) \wedge m_{\mathfrak{P} \cup \mathfrak{P}'}(3)$.

Definition 11. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two picture fuzzy subsets of two near-rings N_1 and N_2 , respectively. Then, the direct product of the picture fuzzy subsets of the near-rings is defined as $\mathfrak{A} \times \mathfrak{B} : N_1 \times N_2 \rightarrow [0, 1]^3$, such that $\mathfrak{A} \times \mathfrak{B} = \{(u, v), (m_{\mathfrak{A} \times \mathfrak{B}}(u, v), a_{\mathfrak{A} \times \mathfrak{B}}(u, v), n_{\mathfrak{A} \times \mathfrak{B}}(u, v)) : u \in N_1 \text{ and}$

$v \in N_2\}$, where $m_{\mathfrak{A} \times \mathfrak{B}}(u, v) = m_{\mathfrak{A}}(u) \wedge m_{\mathfrak{B}}(v)$, $a_{\mathfrak{A} \times \mathfrak{B}}(u, v) = a_{\mathfrak{A}}(u) \vee a_{\mathfrak{B}}(v)$, $n_{\mathfrak{A} \times \mathfrak{B}}(u, v) = n_{\mathfrak{A}}(u) \vee n_{\mathfrak{B}}(v)$.

Similarly, we can define the direct product of n picture fuzzy subsets of the near-rings N_i with $i = 1, \dots, n$.

Remark 4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two picture fuzzy subsets of two near-rings, N_1 and N_2 , respectively. Then, $\mathfrak{A} \times \mathfrak{B}$ is a picture fuzzy ideal of $N_1 \times N_2$ if it satisfies the following conditions:

$$(i) \quad m_{\mathfrak{A} \times \mathfrak{B}}((u_1, u_2) - (v_1, v_2)) \geq m_{\mathfrak{A} \times \mathfrak{B}}((u_1, u_2)) \wedge$$

$$m_{\mathfrak{A} \times \mathfrak{B}}((v_1, v_2)),$$

$$a_{\mathfrak{A} \times \mathfrak{B}}((u_1, u_2) - (v_1, v_2)) \leq a_{\mathfrak{A} \times \mathfrak{B}}((u_1, u_2)) \vee$$

$$a_{\mathfrak{A} \times \mathfrak{B}}((v_1, v_2)),$$

$$n_{\mathfrak{A} \times \mathfrak{B}}((u_1, u_2) - (v_1, v_2)) \leq n_{\mathfrak{A} \times \mathfrak{B}}((u_1, u_2)) \vee$$

$$n_{\mathfrak{A} \times \mathfrak{B}}((v_1, v_2)).$$

$$(ii) \quad m_{\mathfrak{A} \times \mathfrak{B}}((v_1, v_2) + (u_1, u_2) - (v_1, v_2)) \geq m_{\mathfrak{A} \times \mathfrak{B}}((u_1, u_2)),$$

$$a_{\mathfrak{A} \times \mathfrak{B}}((v_1, v_2) + (u_1, u_2) - (v_1, v_2)) \leq a_{\mathfrak{A} \times \mathfrak{B}}((u_1, u_2)),$$

$$n_{\mathfrak{A} \times \mathfrak{B}}((v_1, v_2) + (u_1, u_2) - (v_1, v_2)) \leq n_{\mathfrak{A} \times \mathfrak{B}}((u_1, u_2)).$$

$$(iii) \quad m_{\mathfrak{A} \times \mathfrak{B}}((u_1, u_2)(v_1, v_2)) \geq m_{\mathfrak{A} \times \mathfrak{B}}((v_1, v_2)),$$

$$a_{\mathfrak{A} \times \mathfrak{B}}((u_1, u_2)(v_1, v_2)) \leq a_{\mathfrak{A} \times \mathfrak{B}}((v_1, v_2)),$$

$$n_{\mathfrak{A} \times \mathfrak{B}}((u_1, u_2)(v_1, v_2)) \leq n_{\mathfrak{A} \times \mathfrak{B}}((v_1, v_2)).$$

$$(iv) \quad m_{\mathfrak{A} \times \mathfrak{B}}(((u_1, u_2) + (w_1, w_2))(v_1, v_2) - (u_1, u_2)(v_1, v_2)) \geq$$

$$m_{\mathfrak{A} \times \mathfrak{B}}((w_1, w_2)),$$

$$a_{\mathfrak{A} \times \mathfrak{B}}(((u_1, u_2) + (w_1, w_2))(v_1, v_2) - (u_1, u_2)(v_1, v_2)) \leq$$

$$a_{\mathfrak{A} \times \mathfrak{B}}((w_1, w_2)),$$

$$n_{\mathfrak{A} \times \mathfrak{B}}(((u_1, u_2) + (w_1, w_2))(v_1, v_2) - (u_1, u_2)(v_1, v_2)) \leq$$

$$n_{\mathfrak{A} \times \mathfrak{B}}((w_1, w_2)).$$

Theorem 2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two picture fuzzy ideals of two near-rings N_1 and N_2 , respectively. Then, $\mathfrak{A} \times \mathfrak{B}$ is a picture fuzzy ideal of $N_1 \times N_2$.

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two picture fuzzy ideals of N_1 and N_2 , respectively.

Let $(u_1, u_2), (v_1, v_2), (w_1, w_2) \in N_1 \times N_2$.

$$(i) \quad m_{\mathfrak{A} \times \mathfrak{B}}((u_1, u_2) - (v_1, v_2)) = m_{\mathfrak{A} \times \mathfrak{B}}((u_1 - v_1), (u_2 - v_2)) = m_{\mathfrak{A}}((u_1 - v_1) \wedge m_{\mathfrak{B}}(u_2 - v_2)) \geq \{(m_{\mathfrak{A}}(u_1) \wedge$$

$$m_{\mathfrak{A}}(v_1)) \wedge (m_{\mathfrak{A}}(u_2) \wedge m_{\mathfrak{A}}(v_2))\} = \{(m_{\mathfrak{A}}(u_1) \wedge m_{\mathfrak{B}}(u_2)) \wedge$$

$$(m_{\mathfrak{A}}(v_1) \wedge m_{\mathfrak{B}}(v_2))\} = m_{\mathfrak{A} \times \mathfrak{B}}(u_1, u_2) \wedge m_{\mathfrak{A} \times \mathfrak{B}}(v_1, v_2),$$

$$a_{\mathfrak{A} \times \mathfrak{B}}((u_1, u_2) - (v_1, v_2)) = a_{\mathfrak{A} \times \mathfrak{B}}((u_1 - v_1), (u_2 - v_2)) = a_{\mathfrak{A}}((u_1 - v_1) \vee a_{\mathfrak{B}}(u_2 - v_2)) \leq \{(a_{\mathfrak{A}}(u_1) \vee$$

$$a_{\mathfrak{A}}(v_1)) \vee (a_{\mathfrak{A}}(u_2) \vee a_{\mathfrak{A}}(v_2))\} = \{(a_{\mathfrak{A}}(u_1) \vee a_{\mathfrak{B}}(u_2)) \vee$$

$$(a_{\mathfrak{A}}(v_1) \vee a_{\mathfrak{B}}(v_2))\} = a_{\mathfrak{A} \times \mathfrak{B}}(u_1, u_2) \vee a_{\mathfrak{A} \times \mathfrak{B}}(v_1, v_2),$$

$$n_{\mathfrak{A} \times \mathfrak{B}}((u_1, u_2) - (v_1, v_2)) = n_{\mathfrak{A} \times \mathfrak{B}}((u_1 - v_1), (u_2 - v_2)) = n_{\mathfrak{A}}((u_1 - v_1) \vee n_{\mathfrak{B}}(u_2 - v_2)) \geq \{(n_{\mathfrak{A}}(u_1) \vee$$

$$n_{\mathfrak{A}}(v_1)) \vee (n_{\mathfrak{A}}(u_2) \vee n_{\mathfrak{A}}(v_2))\} = \{(n_{\mathfrak{A}}(u_1) \vee n_{\mathfrak{B}}(u_2)) \vee$$

$$(n_{\mathfrak{A}}(v_1) \vee n_{\mathfrak{B}}(v_2))\} = n_{\mathfrak{A} \times \mathfrak{B}}(u_1, u_2) \vee n_{\mathfrak{A} \times \mathfrak{B}}(v_1, v_2).$$

$$(ii) \quad m_{\mathfrak{A} \times \mathfrak{B}}((v_1, v_2) + (u_1, u_2) - (v_1, v_2)) = m_{\mathfrak{A} \times \mathfrak{B}}((v_1 + u_1 - v_1), (v_2 + u_2 - v_2)) = m_{\mathfrak{A}}((v_1 + u_1 - v_1) \wedge m_{\mathfrak{B}}(v_2 + u_2 - v_2)) \geq m_{\mathfrak{A}}(u_1) \wedge m_{\mathfrak{B}}(u_2) = m_{\mathfrak{A} \times \mathfrak{B}}(u_1, u_2),$$

$$a_{\mathfrak{A} \times \mathfrak{B}}((v_1, v_2) + (u_1, u_2) - (v_1, v_2)) = a_{\mathfrak{A} \times \mathfrak{B}}((v_1 + u_1 - v_1), (v_2 + u_2 - v_2)) = a_{\mathfrak{A}}((v_1 + u_1 - v_1) \vee a_{\mathfrak{B}}(v_2 + u_2 - v_2)) \leq a_{\mathfrak{A}}(u_1) \vee a_{\mathfrak{B}}(u_2) = a_{\mathfrak{A} \times \mathfrak{B}}(u_1, u_2),$$

$$n_{\mathfrak{A} \times \mathfrak{B}}((v_1, v_2) + (u_1, u_2) - (v_1, v_2)) = n_{\mathfrak{A} \times \mathfrak{B}}((v_1 + u_1 - v_1), (v_2 + u_2 - v_2)) = n_{\mathfrak{A}}((v_1 + u_1 - v_1) \vee n_{\mathfrak{B}}(v_2 + u_2 - v_2)) \leq n_{\mathfrak{A}}(u_1) \vee n_{\mathfrak{B}}(u_2) = n_{\mathfrak{A} \times \mathfrak{B}}(u_1, u_2).$$

$$\begin{aligned}
 \text{(iii)} \quad & m_{\mathfrak{A} \times \mathfrak{B}}((u_1, u_2)(v_1, v_2)) = m_{\mathfrak{A} \times \mathfrak{B}}((u_1 v_1), (u_2 v_2)) = \\
 & m_{\mathfrak{A}}((u_1 v_1) \wedge m_{\mathfrak{B}}(u_2 v_2)) \geq (m_{\mathfrak{A}}(v_1) \wedge m_{\mathfrak{B}}(v_2)) = \\
 & m_{\mathfrak{A} \times \mathfrak{B}}(v_1, v_2), \\
 & a_{\mathfrak{A} \times \mathfrak{B}}((u_1, u_2)(v_1, v_2)) = a_{\mathfrak{A} \times \mathfrak{B}}((u_1 v_1), (u_2 v_2)) = \\
 & a_{\mathfrak{A}}((u_1 v_1) \vee a_{\mathfrak{B}}(u_2 v_2)) \leq (a_{\mathfrak{A}}(v_1) \vee a_{\mathfrak{B}}(v_2)) = \\
 & a_{\mathfrak{A} \times \mathfrak{B}}(v_1, v_2), \\
 & n_{\mathfrak{A} \times \mathfrak{B}}((u_1, u_2)(v_1, v_2)) = n_{\mathfrak{A} \times \mathfrak{B}}((u_1 v_1), (u_2 v_2)) = \\
 & n_{\mathfrak{A}}((u_1 v_1) \vee n_{\mathfrak{B}}(u_2 v_2)) \leq (n_{\mathfrak{A}}(v_1) \vee n_{\mathfrak{B}}(v_2)) = \\
 & n_{\mathfrak{A} \times \mathfrak{B}}(v_1, v_2). \\
 \text{(iv)} \quad & m_{\mathfrak{A} \times \mathfrak{B}}(((u_1, u_2) + (w_1, w_2))(v_1, v_2) - (u_1, u_2)(v_1, v_2)) \\
 & = m_{\mathfrak{A} \times \mathfrak{B}}[(((u_1 + w_1)v_1) - u_1 v_1), (((u_2 + w_2)v_2) - \\
 & u_2 v_2)] = m_{\mathfrak{A}}[(((u_1 + w_1)v_1) - u_1 v_1)] \wedge m_{\mathfrak{B}}[(((u_2 + w_2)v_2) - \\
 & u_2 v_2)] \geq m_{\mathfrak{A}}(w_1) \wedge m_{\mathfrak{B}}(w_2) = m_{\mathfrak{A} \times \mathfrak{B}}(w_1, w_2), \\
 & a_{\mathfrak{A} \times \mathfrak{B}}(((u_1, u_2) + (w_1, w_2))(v_1, v_2) - (u_1, u_2)(v_1, v_2)) \\
 & = a_{\mathfrak{A} \times \mathfrak{B}}[(((u_1 + w_1)v_1) - u_1 v_1), (((u_2 + w_2)v_2) - \\
 & u_2 v_2)] = a_{\mathfrak{A}}[(((u_1 + w_1)v_1) - u_1 v_1)] \vee a_{\mathfrak{B}}[(((u_2 + w_2)v_2) - \\
 & u_2 v_2)] \leq a_{\mathfrak{A}}(w_1) \vee a_{\mathfrak{B}}(w_2) = a_{\mathfrak{A} \times \mathfrak{B}}(w_1, w_2), \\
 & n_{\mathfrak{A} \times \mathfrak{B}}(((u_1, u_2) + (w_1, w_2))(v_1, v_2) - (u_1, u_2)(v_1, v_2)) \\
 & = n_{\mathfrak{A} \times \mathfrak{B}}[(((u_1 + w_1)v_1) - u_1 v_1), (((u_2 + w_2)v_2) - \\
 & u_2 v_2)] = n_{\mathfrak{A}}[(((u_1 + w_1)v_1) - u_1 v_1)] \vee n_{\mathfrak{B}}[(((u_2 + w_2)v_2) - \\
 & u_2 v_2)] \leq n_{\mathfrak{A}}(w_1) \vee n_{\mathfrak{B}}(w_2) = n_{\mathfrak{A} \times \mathfrak{B}}(w_1, w_2).
 \end{aligned}$$

Corollary 2. Let n be a positive integer and $\mathfrak{A}_i$ be a picture fuzzy ideal of the near-rings N_i for $i = 1, \dots, n$. Then, $\prod_{i=1}^n \mathfrak{A}_i$ is a picture fuzzy ideal of $\prod_{i=1}^n N_i$.

Proof. The proof follows from Theorem 2 and through induction. $\square$

Example 6. Let $(N_1, +, \cdot)$, $\mathfrak{P}_1 = (m_{\mathfrak{P}_1}, a_{\mathfrak{P}_1}, n_{\mathfrak{P}_1})$, and $(\mathbb{Z}, +, \cdot)$, $\mathfrak{P} = (m_{\mathfrak{P}}, a_{\mathfrak{P}}, n_{\mathfrak{P}})$ be the near-rings with their corresponding PFS presented in Examples 1 and 2, respectively. By defining the picture fuzzy set $\mathfrak{P}_1 \times \mathfrak{P}$ on $N_1 \times \mathbb{Z}$ as follows:

$$\begin{aligned}
 m_{\mathfrak{P}_1 \times \mathfrak{P}}((x, y)) &= \begin{cases} 0.8 & \text{if } x = 0 \text{ and } y \text{ is a multiple of 4;} \\ 0.4 & \text{if } x \neq 0 \text{ and } y \text{ is a multiple of 4;} \\ 0.2 & \text{otherwise,} \end{cases} \\
 a_{\mathfrak{P}_1 \times \mathfrak{P}}((x, y)) &= \begin{cases} 0.15 & \text{if } x = 0 \text{ and } y \text{ is a multiple of 4;} \\ 0.2 & \text{if } x \neq 0 \text{ and } y \text{ is a multiple of 4;} \\ 0.35 & \text{otherwise,} \end{cases}
 \end{aligned}$$

and

$$n_{\mathfrak{P}_1 \times \mathfrak{P}}((x, y)) = \begin{cases} 0.05 & \text{if } x = 0 \text{ and } y \text{ is a multiple of 4;} \\ 0.4 & \text{if } y \text{ is not a multiple of 4;} \\ 0.3 & \text{if } x \neq 0 \text{ and } y \text{ is a multiple of 4,} \end{cases}$$

we obtain through Theorem 2 that $\mathfrak{P}_1 \times \mathfrak{P}$ is a picture fuzzy

ideal of $N_1 \times \mathbb{Z}$.

3.3 Homomorphism of Near-Rings through Picture Fuzzy Ideals

We study picture fuzzy ideals under near-ring homomorphisms.

Definition 12. A map $g : N_1 \rightarrow N_2$, where N_1 and N_2 are near-rings, is said to be a homomorphism if $g(u + v) = g(u) + g(v)$ and $g(uv) = g(u)g(v)$, for all $u, v \in N_1$.

Let $g : N_1 \rightarrow N_2$ be a map, where N_1 and N_2 are near-rings. For any picture fuzzy set $\mathfrak{A} = (m_{\mathfrak{A}}, a_{\mathfrak{A}}, n_{\mathfrak{A}})$ in N_2 , we define the picture fuzzy set $\mathfrak{A}^g = (m_{\mathfrak{A}}^g, a_{\mathfrak{A}}^g, n_{\mathfrak{A}}^g)$ in N_1 by $m_{\mathfrak{A}}^g(u) = m_{\mathfrak{A}}(g(u))$, $a_{\mathfrak{A}}^g(u) = a_{\mathfrak{A}}(g(u))$, $n_{\mathfrak{A}}^g(u) = n_{\mathfrak{A}}(g(u))$, for all $u \in N_1$.

Theorem 3. Let $g : N_1 \rightarrow N_2$ be a homomorphism, where N_1 and N_2 are near-rings. If the picture fuzzy set $\mathfrak{A} = (m_{\mathfrak{A}}, a_{\mathfrak{A}}, n_{\mathfrak{A}})$ in N_2 is a picture fuzzy ideal of N_2 , then the picture fuzzy set $\mathfrak{A}^g = (m_{\mathfrak{A}}^g, a_{\mathfrak{A}}^g, n_{\mathfrak{A}}^g)$ in N_1 is a picture fuzzy ideal of N_1 .

Proof. Because g is a homomorphism, it follows that for any $u, v, w \in N_1$, $g(u - v) = g(u) - g(v)$, $g(u + v - u) = g(v) + g(u) - g(v)$, $g(uv) = g(u)g(v)$, and $g((u + w)v - uv) = g(u)g(v) + g(w)g(v) - g(u)g(v)$, we obtain the following.

$$\begin{aligned}
 \text{(i)} \quad & m_{\mathfrak{A}}^g(u - v) = m_{\mathfrak{A}}(g(u) - g(v)) \geq m_{\mathfrak{A}}(g(u)) \wedge m_{\mathfrak{A}}(g(v)) \\
 & = m_{\mathfrak{A}}^g(u) \wedge m_{\mathfrak{A}}^g(v), \\
 & a_{\mathfrak{A}}^g(u - v) = a_{\mathfrak{A}}(g(u) - g(v)) \leq a_{\mathfrak{A}}(g(u)) \vee a_{\mathfrak{A}}(g(v)) = \\
 & a_{\mathfrak{A}}^g(u) \vee a_{\mathfrak{A}}^g(v), \\
 & n_{\mathfrak{A}}^g(u - v) = n_{\mathfrak{A}}(g(u) - g(v)) \leq n_{\mathfrak{A}}(g(u)) \vee n_{\mathfrak{A}}(g(v)) = \\
 & n_{\mathfrak{A}}^g(u) \vee n_{\mathfrak{A}}^g(v). \\
 \text{(ii)} \quad & m_{\mathfrak{A}}^g(v + u - v) = m_{\mathfrak{A}}(g(v) + g(u) - g(v)) \geq m_{\mathfrak{A}}(g(u)) = \\
 & m_{\mathfrak{A}}^g(u), \\
 & a_{\mathfrak{A}}^g(v + u - v) = a_{\mathfrak{A}}(g(v) + g(u) - g(v)) \leq a_{\mathfrak{A}}(g(u)) = \\
 & a_{\mathfrak{A}}^g(u), \\
 & n_{\mathfrak{A}}^g(v + u - v) = n_{\mathfrak{A}}(g(v) + g(u) - g(v)) \leq n_{\mathfrak{A}}(g(u)) = \\
 & n_{\mathfrak{A}}^g(u). \\
 \text{(iii)} \quad & m_{\mathfrak{A}}^g(uv) = m_{\mathfrak{A}}(g(u)g(v)) \geq m_{\mathfrak{A}}(g(v)) = m_{\mathfrak{A}}^g(v), \\
 & a_{\mathfrak{A}}^g(uv) = a_{\mathfrak{A}}(g(u)g(v)) \leq a_{\mathfrak{A}}(g(v)) = a_{\mathfrak{A}}^g(v), \\
 & n_{\mathfrak{A}}^g(uv) = n_{\mathfrak{A}}(g(u)g(v)) \leq n_{\mathfrak{A}}(g(v)) = n_{\mathfrak{A}}^g(v). \\
 \text{(iv)} \quad & m_{\mathfrak{A}}^g((u + w)v - uv) = m_{\mathfrak{A}}(g((u + w)v - uv)) = m_{\mathfrak{A}}((g(u) \\
 & + g(w))g(v) - g(u)g(v)) \geq m_{\mathfrak{A}}(g(w)) = m_{\mathfrak{A}}^g(w), \\
 & a_{\mathfrak{A}}^g((u + w)v - uv) = a_{\mathfrak{A}}(g((u + w)v - uv)) = a_{\mathfrak{A}}((g(u) + \\
 & g(w))g(v) - g(u)g(v)) \leq a_{\mathfrak{A}}(g(w)) = a_{\mathfrak{A}}^g(w), \\
 & n_{\mathfrak{A}}^g((u + w)v - uv) = n_{\mathfrak{A}}(g((u + w)v - uv)) = n_{\mathfrak{A}}((g(u) + \\
 & g(w))g(v) - g(u)g(v)) \leq n_{\mathfrak{A}}(g(w)) = n_{\mathfrak{A}}^g(w).
 \end{aligned}$$

Example 7. Let $(\mathbb{Z}, +, \cdot)$, $\mathfrak{P} = (m_{\mathfrak{P}}, a_{\mathfrak{P}}, n_{\mathfrak{P}}) \in N$ be the near-ring and picture fuzzy sets on $\mathbb{Z}$ presented in Example 1. Let $(2\mathbb{Z}, +, \cdot)$ be the near-ring under a standard addition of even integers and “ $\cdot$ ” be defined as $a \cdot b = b$ for all $a, b \in 2\mathbb{Z}$, $g : 2\mathbb{Z} \rightarrow \mathbb{Z}$ is the near-ring homomorphism defined through the inclusion map. Theorem 3 asserts that the picture fuzzy set $\mathfrak{P}^g = (m_{\mathfrak{P}^g}, a_{\mathfrak{P}^g}, n_{\mathfrak{P}^g})$ in $2\mathbb{Z}$ is a picture fuzzy ideal of $2\mathbb{Z}$.

4. Conclusion

This paper contributes to the study of picture fuzzy algebraic structures by discussing the picture fuzzy ideals of near-rings. Several interesting properties, propositions, and theorems were investigated, and some examples are shown through a picture fuzzy environment. Because picture fuzzy sets are a generalization of intuitionistic fuzzy and fuzzy sets, the results in this study can be considered a generalization of picture fuzzy near-rings.

As future research, it will be interesting to study picture fuzzy quasi-ideals, picture fuzzy bi-ideals in near-rings, rough picture fuzzy near-rings, and soft picture fuzzy near-rings. We can also elaborate more results on picture fuzzy ideals for a finite number of near-rings.

Conflict of Interest

No potential conflicts of interest relevant to this article are reported.

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