Orthogonal F -Contraction Mapping on O -Complete Metric Space with Applications

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Abstract

In this paper, we introduce orthogonal concepts concerning F -contraction mappings and demonstrate some fixed-point theorems for self-mapping in a complete orthogonal metric space. Some well-known results in the literature are generalized and modified based on the demonstrated results. An example is provided to support our results, which are used in an application.

Keywords: Orthogonal set, Orthogonal complete metric space, Orthogonal continuous, Orthogonal preserving, Orthogonal F -contraction, Fixed point

1. Introduction

One of the most important results of a mathematical analysis is the famous fixed-point result, called Banach contraction theory. In several branches of mathematics, this is the most commonly used fixed-point result, and has been generalized in many different directions. One natural way to reinforce the Banach contraction concept is to replace the metric space with other generalized metric spaces. The fixed-point result in the setting of complete metric spaces was established by Wardowski [1], which is a generalization of the Banach contraction principle in metric spaces. However, in many branches of mathematics, the notion of an orthogonal set has many applications and has several methods of orthogonality. Gordji et al. [2] introduced the current concept of orthogonality in metric spaces and demonstrated the fixed-point result for contraction mappings in metric spaces equipped with the new orthogonality. In a generalized orthogonal metric space, Gordji and Habibi [3] proved the theory of fixed points. Sawangsup et al. [4] introduced the new concept of orthogonal F -contraction mappings and fixed-point theorems on orthogonal-complete metric space were proven. Orthogonal contractive-type mappings have been studied by many authors, and important results were obtained in [5–9].

In Section 2, for the subsequent usage, we remember some essential notes and definitions from the current literature. In Section 3, in an orthogonal complete metric space, we introduce a new F -contraction mapping and prove some fixed-point theorems for this contraction mapping. We apply our results to demonstrate the presence of a unique solution to the ordinary

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differential equation in Section 4.

2. Preliminaries

In this study, Ξ , $\mathbb{R}^+$, and $\mathbb{N}$ denote the non-empty set, positive real numbers, and positive integer sets, respectively.

First, we recall the notion of an auxiliary function, introduced in 2012 by Wardowski [1], as follows:

Definition 2.1 [1]. Let (Ξ, d) be a metric space. A mapping $\Omega : \Xi \rightarrow \Xi$ is said to be an F -contraction if $\iota > 0$ and $F \in \Upsilon$ such that

$$\iota + F(d(\Omega\varpi, \Omega\rho)) \leq F(d(\varpi, \rho))$$

holds for any $\varpi, \rho \in \Xi$ with $d(\Omega\varpi, \Omega\rho) > 0$, where Υ is the set of all functions $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfying the following properties:

(F_1) F is strictly increasing;

(F_2) for each sequence $\{\varepsilon_m\}$ of positive numbers, we have

$$\lim_{m \rightarrow \infty} \varepsilon_m = 0 \text{ iff } \lim_{m \rightarrow \infty} F(\varepsilon_m) = -\infty;$$

(F_3) $\exists q \in (0, 1) \ni \lim_{\varepsilon \rightarrow 0^+} \varepsilon^q F(\varepsilon) = 0$.

In 2013, Secelean [10] demonstrated that, in Definition 2.1, the condition (F_2) can be altered. The definition of an F -contraction of the Hardy-Rogers-type was introduced by Cosentino and Vetro [11]. In 2014, Piri and Kumam [12] demonstrated that in Definition 2.1, the condition (F_3) can be altered. Later, in 2020, Popescu and Stan [13] proved some fixed point theorem concerning an F -contraction in a complete metric space as follows:

Theorem 2.1 [13]. Let Ω be a self-mapping of a complete metric space Ξ . Suppose that there exists $\iota > 0$ such that $\forall \varpi, \rho \in \Xi$, $d(\Omega\varpi, \Omega\rho) > 0$,

$$\begin{aligned} \Rightarrow \iota + F(d(\Omega\varpi, \Omega\rho)) &\leq F(\theta_1 d(\varpi, \rho) + \theta_2 d(\varpi, \Omega\varpi) \\ &\quad + \theta_3 d(\rho, \Omega\rho) + \theta_4 d(\varpi, \Omega\rho) \\ &\quad + \theta_5 d(\rho, \Omega\varpi)), \end{aligned}$$

where $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is an increasing mapping, $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \geq 0$, $\theta_4 < \frac{1}{2}$, $\theta_3 < 1$, $\theta_1 + \theta_2 + \theta_3 + 2\theta_4 = 1$, $0 < \theta_1 + \theta_4 + \theta_5 \leq 1$. Then, Ω has a unique fixed point $\varpi^* \in \Xi$, and for every $\varpi \in \Xi$, the sequence $\{\Omega^m \varpi\}_{m \in \mathbb{N}}$ converges to ϖ^* .

By contrast, the definition of an orthogonal set (or O set) and the properties of orthogonal sets were introduced by Gordji et al. [2] as follows:

Definition 2.2 [2]. Let $\Xi \neq \phi$ and $\perp \subseteq \Xi \times \Xi$ be a binary relation. If $\perp$ satisfies the following consecutive conditions,

$$\exists \varpi_0 \in \Xi : (\forall \varpi \in \Xi, \varpi \perp \varpi_0) \quad \text{or} \quad (\forall \varpi \in \Xi, \varpi_0 \perp \varpi),$$

then it is said to be an O set. We indicate this O set by $(\Xi, \perp)$.

Example 2.2 [2]. Let $\Xi = [0, \infty)$ and define $\varpi \perp \rho$ if $\varpi \rho \in \{\varpi, \rho\}$. Then, by setting $\varpi_0 = 0$ or $\varpi_0 = 1$, $(\Xi, \perp)$ is an O -set.

Now, we present the concepts of an an orthogonal sequence (O -sequence), an O -complete orthogonal metric space, and a $\perp$ -preserving mapping.

Definition 2.3 [2]. Let $(\Xi, \perp)$ be an O set. A sequence $\{\varpi_m\}$ is said to be an O -sequence if

$$(\forall m \in \mathbb{N}, \varpi_m \perp \varpi_{m+1}) \quad \text{or} \quad (\forall m \in \mathbb{N}, \varpi_{m+1} \perp \varpi_m).$$

Definition 2.4 [2]. The triplet $(\Xi, \perp, d)$ is considered to be an orthogonal metric space if $(\Xi, \perp)$ is an O set, and (Ξ, d) is a metric space.

Definition 2.5 [2]. Let $(\Xi, \perp)$ be an O set. A mapping $\Omega : \Xi \rightarrow \Xi$ is called $\perp$ -preserving if $\Omega\varpi \perp \Omega\rho$ whenever $\varpi \perp \rho$.

In this study, we adapt the concepts of F -contraction mapping to orthogonal sets and prove some fixed-point theorems for an F -contraction mapping in the complete orthogonality of the metric space.

3. Main Results

In this section, we introduce a new F -contraction mapping, inspired by the concepts of F -contraction mapping, described by Popescu and Stan [13], and demonstrate some fixed-point theorems in an orthogonal-complete metric space for this contraction mapping.

Theorem 3.1. Let $(\Xi, \perp, d)$ be an orthogonal-complete metric space, and an orthogonal element ϖ_0 . Let Ω be a self-mapping of Ξ . Suppose that $\iota > 0$ exists such that

$$\forall \varpi, \rho \in \Xi \text{ with } \varpi \perp \rho [d(\Omega\varpi, \Omega\rho) > 0 \quad (1)$$

$$\begin{aligned} \Rightarrow \iota + F(d(\Omega\varpi, \Omega\rho)) &\leq F(\theta_1 d(\varpi, \rho)) \\ &\quad + \theta_2 d(\varpi, \Omega\varpi) + \theta_3 d(\varpi, \Omega\rho) \\ &\quad + \theta_4 d(\varpi, \Omega\rho) + \theta_5 d(\rho, \Omega\varpi)], \end{aligned} \quad (2)$$

where $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is an increasing mapping; $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ are non-negative numbers; and $\theta_4 < \frac{1}{2}, \theta_3 < 1, \theta_1 + \theta_2 + \theta_3 + 2\theta_4 = 1, 0 < \theta_1 + \theta_4 + \theta_5 \leq 1$, where Ω is $\perp$ -preserving. Then, Ω has a unique fixed-point $t \in \Xi$, and for every $\varpi \in \Xi$, the sequence $\{\Omega^m \varpi\}_{m \in \mathbb{N}}$ converges to t .

Proof. Because $(\Xi, \perp)$ is an O -set,

$$\exists \varpi_0 \in \Xi : (\forall \varpi \in \Xi, \varpi \perp \varpi_0) \quad \text{or} \quad (\forall \varpi \in \Xi, \varpi_0 \perp \varpi).$$

It follows that $\varpi_0 \perp \Omega \varpi_0$ or $\Omega \varpi_0 \perp \varpi_0$. Let

$$\varpi_1 := \Omega \varpi_0, \dots, \varpi_{m+1} := \Omega \varpi_m = \Omega^{m+1} \varpi_0, \quad (3)$$

$\forall m \in \mathbb{N} \cup \{0\}$. If $\varpi_m = \varpi_{m+1}$ for any $m \in \mathbb{N} \cup \{0\}$, it is then evident that ϖ_m is a fixed-point of Ω . We assume that $\varpi_m \neq \varpi_{m+1} \forall m \in \mathbb{N} \cup \{0\}$. Thus, we derive $d(\Omega \varpi_m, \Omega \varpi_{m+1}) > 0 \forall m \in \mathbb{N} \cup \{0\}$. Because Ω is $\perp$ -preserving, we obtain

$$\varpi_m \perp \varpi_{m+1} \quad \text{or} \quad \varpi_{m+1} \perp \varpi_m,$$

$\forall m \in \mathbb{N} \cup \{0\}$. This means that $\{\varpi_m\}$ is an O -sequence. Hence, we presume that

$$0 < d(\varpi_m, \Omega \varpi_m) = d(\Omega \varpi_{m-1}, \Omega \varpi_m), \quad \forall m \in \mathbb{N}. \quad (4)$$

Now, let $d_m = d(\varpi_m, \varpi_{m+1})$. Through the monotony of F , we derive $\forall m \in \mathbb{N}$

$$\begin{aligned} & \iota + F(d_m) \\ &= \iota + F(d(\varpi_m, \varpi_{m+1})) = \iota + F(d(\Omega \varpi_{m-1}, \Omega \varpi_m)) \\ &\leq F(\theta_1 d(\varpi_{m-1}, \varpi_m) + \theta_2 d(\varpi_{m-1}, \Omega \varpi_{m-1}) \\ &\quad + \theta_3 d(\varpi_m, \Omega \varpi_m) + \theta_4 d(\varpi_{m-1}, \Omega \varpi_m) \\ &\quad + \theta_5 d(\varpi_m, \Omega \varpi_{m-1})) \\ &= F(\theta_1 d(\varpi_{m-1}, \varpi_m) + \theta_2 d(\varpi_{m-1}, \varpi_m) \\ &\quad + \theta_3 d(\varpi_m, \varpi_{m+1}) + \theta_4 d(\varpi_{m-1}, \varpi_{m+1}) \\ &\quad + \theta_5 d(\varpi_m, \varpi_m)) \\ &= F(\theta_1 d_{m-1} + \theta_2 d_{m-1} + \theta_3 d_m + \theta_4 d(\varpi_{m-1}, \varpi_{m+1})) \\ &\leq F(\theta_1 d_{m-1} + \theta_2 d_{m-1} + \theta_3 d_m + \theta_4 (d_{m-1} + d_m)) \\ &= F((\theta_1 + \theta_2 + \theta_4) d_{m-1} + (\theta_3 + \theta_4) d_m). \end{aligned}$$

It proceeds that

$$\begin{aligned} F(d_m) &\leq F((\theta_1 + \theta_2 + \theta_4) d_{m-1} + (\theta_3 + \theta_4) d_m) - \iota \\ &< F((\theta_1 + \theta_2 + \theta_4) d_{m-1} + (\theta_3 + \theta_4) d_m). \end{aligned} \quad (5)$$

Thus, from the monotony of F , we derive

$$d_m < (\theta_1 + \theta_2 + \theta_4) d_{m-1} + (\theta_3 + \theta_4) d_m,$$

and hence

$$(1 - \theta_3 - \theta_4) d_m < (\theta_1 + \theta_2 + \theta_4) d_{m-1},$$

$\forall m \in \mathbb{N}$. Because $\theta_3 \neq 1$ and $\theta_1 + \theta_2 + \theta_3 + 2\theta_4 = 1$, we conclude that $1 - \theta_3 - \theta_4 > 0$, and thus

$$d_m < \frac{\theta_1 + \theta_2 + \theta_4}{1 - \theta_3 - \theta_4} d_{m-1} = d_{m-1},$$

$\forall m \in \mathbb{N}$. Therefore, the sequence $\{d_m\}_{m \in \mathbb{N}}$ is strictly decreasing, and thus, $\lim_{m \rightarrow \infty} d_m = d$. Suppose that $d > 0$. Because F is an increasing mapping, $\lim_{\varpi \rightarrow d} F(\varpi) = F(d + 0)$. Because $m \rightarrow \infty$ in Eq. (5), we obtain $F(d + 0) \leq F(d + 0) - \iota$. Therefore,

$$\lim_{m \rightarrow \infty} d_m = 0. \quad (6)$$

Next, we prove that $\{\varpi_m\}_{m \in \mathbb{N}}$ is an O -Cauchy sequence. Suppose that $\epsilon > 0$ and sequences $\{i(m)\}_{m \in \mathbb{N}}$ and $\{j(m)\}_{m \in \mathbb{N}}$ of natural numbers exist such that $i(m) > j(m) > m$,

$$d(\varpi_{i(m)}, \varpi_{j(m)}) > \epsilon, \quad d(\varpi_{i(m)-1}, \varpi_{j(m)}) \leq \epsilon, \quad \forall m \in \mathbb{N}. \quad (7)$$

We derive

$$\begin{aligned} \epsilon &< d(\varpi_{i(m)}, \varpi_{j(m)}) \leq d(\varpi_{i(m)}, \varpi_{i(m)-1}) \\ &\quad + d(\varpi_{i(m)-1}, \varpi_{j(m)}) \\ &\leq d(\varpi_{i(m)-1}, \varpi_{j(m)}) + \epsilon. \end{aligned}$$

From Eq. (6), we obtain

$$\lim_{m \rightarrow \infty} d(\varpi_{i(m)}, \varpi_{j(m)}) = \epsilon. \quad (8)$$

Because $d(\varpi_{i(m)}, \varpi_{j(m)}) > \epsilon > 0$ and the monotony of F , we obtain

$$\begin{aligned} & \iota + F(d(\varpi_{i(m)}, \varpi_{j(m)})) \\ &\leq F(\theta_1 d(\varpi_{i(m)-1}, \varpi_{j(m)-1}) + \theta_2 d(\varpi_{i(m)-1}, \Omega \varpi_{i(m)-1}) \\ &\quad + \theta_3 d(\varpi_{j(m)-1}, \Omega \varpi_{j(m)-1}) + \theta_4 d(\varpi_{i(m)-1}, \Omega \varpi_{j(m)-1}) \\ &\quad + \theta_5 d(\Omega \varpi_{i(m)-1}, \varpi_{j(m)-1})) \\ &= F(\theta_1 d(\varpi_{i(m)-1}, \varpi_{j(m)-1}) + \theta_2 d(\varpi_{i(m)-1}, \varpi_{i(m)}) \\ &\quad + \theta_3 d(\varpi_{j(m)-1}, \varpi_{j(m)}) + \theta_4 d(\varpi_{i(m)-1}, \varpi_{j(m)}) \end{aligned}$$

$$\begin{aligned}
 & + \theta_5 d(\varpi_{i(m)}, \varpi_{j(m)-1})) \\
 \leq & F(\theta_1 [d(\varpi_{i(m)}, \varpi_{j(m)}) + d_{i(m)-1} + d_{j(m)-1}] \\
 & + \theta_2 d_{i(m)-1} + \theta_3 d_{j(m)-1} \\
 & + \theta_4 [d(\varpi_{i(m)}, \varpi_{j(m)}) + d_{i(m)-1}] \\
 & + \theta_5 [d(\varpi_{i(m)}, \varpi_{j(m)}) + d_{j(m)-1}]) \\
 = & F((\theta_1 + \theta_4 + \theta_5) d(\varpi_{i(m)}, \varpi_{j(m)}) + (\theta_1 + \theta_2 \\
 & + \theta_4) d_{i(m)-1} + (\theta_1 + \theta_3 + \theta_5) d_{j(m)-1}) \\
 \leq & F(d(\varpi_{i(m)}, \varpi_{j(m)}) + d_{i(m)-1} \\
 & + (\theta_1 + \theta_3 + \theta_5) d_{j(m)-1}).
 \end{aligned}$$

Because $m \rightarrow \infty$, we derive $\iota + F(\epsilon + 0) \leq F(\epsilon + 0)$. Therefore, $\{\varpi_m\}_{m \in \mathbb{N}}$ is an O -Cauchy sequence. Because (Ξ, d) is an O -complete metric space, we find that $\{\varpi_m\}_{m \in \mathbb{N}}$ converges to a certain point t in Ξ . If there exists a sequence $\{i(m)\}_{m \in \mathbb{N}}$ of natural numbers such that $\varpi_{i(m)+1} = \Omega \varpi_{i(m)} = \Omega t$, then $\lim_{m \rightarrow \infty} \varpi_{i(m)+1} = t$, and $\Omega t = t$. Otherwise, there exists $M \in \mathbb{N}$ such that $\varpi_{m+1} = \Omega \varpi_m \neq \Omega t, \forall m \geq M$. We assume that $\Omega t \neq t$. From Eq. (2), we obtain the following:

$$\begin{aligned}
 & \iota + F(d(\Omega \varpi_m, \Omega t)) \\
 \leq & F(\theta_1 d(\varpi_m, t) + \theta_2 d(\varpi_m, \Omega \varpi_m) \\
 & + \theta_3 d(t, \Omega t) + \theta_4 d(\varpi_m, \Omega t) + \theta_5 d(\Omega \varpi_m, t)),
 \end{aligned}$$

and thus

$$\begin{aligned}
 & \iota + F(d(\Omega \varpi_m, \Omega t)) \\
 = & F(\theta_1 d(\varpi_m, t) + \theta_2 d(\varpi_m, \varpi_{m+1}) + \theta_3 d(t, \Omega t) \\
 & + \theta_4 d(\varpi_m, \Omega t) + \theta_5 d(\varpi_{m+1}, t)).
 \end{aligned}$$

Because F is increasing, we derive

$$\begin{aligned}
 d(\Omega \varpi_m, \Omega t) & < \theta_1 d(\varpi_m, t) + \theta_2 d(\varpi_m, \varpi_{m+1}) \\
 & + \theta_3 d(t, \Omega t) + \theta_4 d(\varpi_m, \Omega t) \\
 & + \theta_5 d(\varpi_{m+1}, t).
 \end{aligned}$$

Because $m \rightarrow \infty$, we have

$$d(t, \Omega t) \leq \theta_3 d(t, \Omega t) + \theta_4 d(t, \Omega t) < d(t, \Omega t).$$

Therefore, $\Omega t = t$. Let $t, r \in \Xi$ be two distinct fixed-points of Ω . Thus, $\Omega t = t \neq r = \Omega r$. Hence, $d(\Omega t, \Omega r) = d(t, r) > 0$. Because $0 < \theta_1 + \theta_4 + \theta_5 \leq 1$, we obtain

$$\iota + F(d(t, r)) = \iota + F(d(\Omega t, \Omega r))$$

$$\begin{aligned}
 & \leq F(\theta_1 d(t, r) + \theta_2 d(t, \Omega t) + \theta_3 d(r, \Omega r) \\
 & + \theta_4 d(t, \Omega r) + \theta_5 d(r, \Omega t)) \\
 = & F(\theta_1 d(t, r) + \theta_4 d(t, r) + \theta_5 d(t, r)) \\
 \leq & F((\theta_1 + \theta_4 + \theta_5) d(t, r)) \\
 \leq & F(d(t, r)).
 \end{aligned}$$

Therefore, Ω has a unique fixed point. $\square$

Corollary 3.2. Let $(\Xi, \perp, d)$ be an orthogonal-complete metric space and an orthogonal element ϖ_0 . Let Ω be a self-mapping of Ξ . Presume that $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is an increasing mapping, and $\iota > 0$ such that

$$\begin{aligned}
 & \forall \varpi, \varrho \in \Xi \text{ with } \varpi \perp \varrho [d(\Omega \varpi, \Omega \varrho) > 0 \\
 & \Rightarrow \iota + F(d(\Omega \varpi, \Omega \varrho)) \leq F(d(\varpi, \varrho)),
 \end{aligned}$$

for all $\varpi, \varrho \in \Xi$, and $\Omega \varpi \neq \Omega \varrho$, where Ω is $\perp$ -preserving. Then, Ω has a unique fixed-point in Ξ .

Example 3.3. Let $\Xi = [0, 1] \cup [2, 3]$ and $d : \Xi \times \Xi \rightarrow [0, \infty)$ be a mapping defined by

$$d(\varpi, \varrho) = \max\{\varpi, \varrho\},$$

for all $\varpi, \varrho \in \Xi$. We define the binary relation $\perp$ on Ξ by $\varpi \perp \varrho$ if $\varpi \varrho \leq (\varpi \vee \varrho)$, where $\varpi \vee \varrho = \varpi$ or ϱ . Then, (Ξ, d) is an O -complete metric space. The mapping $\Omega : \Xi \rightarrow \Xi$ is defined as

$$\Omega \varrho = \begin{cases} 1, & \text{if } \varrho \in [0, 1], \\ \frac{1}{\varrho}, & \text{if } \varrho \in [2, 3]. \end{cases}$$

Clearly, Ω is $\perp$ -preserving. Now, let us consider mapping F defined by $F(t) = \ln t$. Let $\varpi \perp \varrho$ and $\iota > 0$. Without a loss of generality, we may assume that $\varpi \varrho \leq \varpi$. Note that if $d(\Omega \varpi, \Omega \varrho) > 0$, then

$$\iota + F(d(\Omega \varpi, \Omega \varrho)) \leq F(d(\varpi, \varrho)), \quad \forall \varpi, \varrho \in \Xi$$

is equivalent to

$$d(\Omega \varpi, \Omega \varrho) \leq e^{-\iota} (d(\varpi, \varrho)), \quad \forall \varpi, \varrho \in \Xi. \quad (9)$$

We are now considering the following cases:

Case 1: Letting $\varpi = 0$ and $\varrho \in [0, 1]$, then

$$d(\Omega \varpi, \Omega \varrho) = 1 \text{ and } d(\varpi, \varrho) = 1.$$

It is obvious that Eq. (9) is satisfied.

Case 2: Letting $\varpi = 0$ and $\varrho \in [2, 3]$, then

$$d(\Omega\varpi, \Omega\varrho) = 1 \text{ and } d(\varpi, \varrho) = \varrho.$$

It is clear that Eq. (9) is satisfied.

Case 3: Letting $\varpi \in [0, 1]$ and $\varrho \in (0, 1]$, then

$$d(\Omega\varpi, \Omega\varrho) = 1 \text{ and } d(\varpi, \varrho) = 1.$$

It is obvious that Eq. (9) is satisfied.

Case 4: Letting $\varpi \in [2, 3]$ and $\varrho \in [0, 1]$, then

$$d(\Omega\varpi, \Omega\varrho) = 1 \text{ and } d(\varpi, \varrho) = 1.$$

It is obvious that Eq. (9) is satisfied.

Therefore, all conditions of Corollary (3.2) are satisfied. Hence, Ω has a unique fixed point in Ξ , that is, a point $\varpi = 1$.

Theorem 3.4. Let Ω be a self-mapping of an orthogonal-complete metric space Ξ . Suppose that $\iota > 0$ exists such that

$$\begin{aligned} \forall \varpi, \varrho \in \Xi \text{ with } \varpi \perp \varrho [d(\Omega\varpi, \Omega\varrho) > 0 \\ \Rightarrow \iota + F(d(\Omega\varpi, \Omega\varrho)) \leq F(d(\varpi, \varrho))], \end{aligned} \quad (10)$$

where $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a mapping that satisfies the conditions (F_2) and (F_3'') , where (F_3'') F is continuous on $(0, \alpha)$, where α is a positive real number, and Ω is $\perp$ -preserving. Then, Ω has a unique fixed-point $t \in \Xi$, and for every $\varpi \in \Xi$, the sequence $\{\Omega^m \varpi\}_{m \in \mathbb{N}}$ converges to t .

Proof. Because $(\Xi, \perp)$ is an O -set,

$$\exists \varpi_0 \in \Xi : (\forall \varpi \in \Xi, \varpi \perp \varpi_0) \text{ or } (\forall \varpi \in \Xi, \varpi_0 \perp \varpi).$$

It follows that $\varpi_0 \perp \Omega\varpi_0$ or $\Omega\varpi_0 \perp \varpi_0$. Let

$$\varpi_1 := \Omega\varpi_0, \dots, \varpi_{m+1} := \Omega\varpi_m = \Omega^{m+1}\varpi_0, \quad (11)$$

for all $m \in \mathbb{N} \cup \{0\}$. If $\varpi_m = \varpi_{m+1}$ for any $m \in \mathbb{N} \cup \{0\}$, then it is evident that ϖ_m is a fixed point of Ω . We assume that $\varpi_m \neq \varpi_{m+1}$ for all $m \in \mathbb{N} \cup \{0\}$. Thus, we derive $d(\Omega\varpi_m, \Omega\varpi_{m+1}) > 0$ for all $m \in \mathbb{N} \cup \{0\}$. Because Ω is $\perp$ -preserving, we have the following:

$$\varpi_m \perp \varpi_{m+1} \text{ or } \varpi_{m+1} \perp \varpi_m,$$

for all $m \in \mathbb{N} \cup \{0\}$. This means that $\{\varpi_m\}$ is an O -sequence. Hence, we presume that

$$0 < d(\varpi_m, \Omega\varpi_m) = d(\Omega\varpi_{m-1}, \Omega\varpi_m), \quad \forall m \in \mathbb{N} \quad (12)$$

From Eq. (10), we obtain

$$\iota + F(d(\Omega\varpi_{m-1}, \Omega\varpi_m)) \leq F(d(\varpi_{m-1}, \varpi_m)), \quad \forall m \in \mathbb{N}. \quad (13)$$

That is

$$\begin{aligned} \iota + F(d(\Omega\varpi_{m-1}, \Omega\varpi_m)) &\leq F(d(\varpi_{m-1}, \varpi_m)) - \iota \\ &= F(d(\Omega\varpi_{m-2}, \Omega\varpi_{m-1})) - \iota \\ &\leq F(d(\varpi_{m-2}, \varpi_{m-1})) - 2\iota \\ &= F(d(\Omega\varpi_{m-3}, \Omega\varpi_{m-2})) - 2\iota \\ &\leq F(d(\varpi_{m-3}, \varpi_{m-2})) - 3\iota \\ &= F(d(\Omega\varpi_{m-4}, \Omega\varpi_{m-3})) - 3\iota \\ &\leq \dots \\ &\leq F(d(\varpi_0, \varpi_1)) - m\iota. \end{aligned}$$

This means that

$$\lim_{m \rightarrow \infty} F(d(\varpi_m, \varpi_{m+1})) = \lim_{m \rightarrow \infty} F(d(\Omega\varpi_{m-1}, \Omega\varpi_m)) = -\infty.$$

By (F_2) , we derive the following:

$$\lim_{n \rightarrow \infty} F(d(\varpi_m, \varpi_{m+1})) = 0. \quad (14)$$

Next, we prove that $\{\varpi_m\}_{m \in \mathbb{N}}$ is an O -Cauchy sequence. Suppose that $0 < \epsilon < \alpha$ and sequences $\{i(m)\}_{m \in \mathbb{N}}$ and $\{j(m)\}_{m \in \mathbb{N}}$ of natural numbers exist such that

$$\begin{aligned} i(m) > j(m) > m, \quad d(\varpi_{i(m)}, \varpi_{j(m)}) \geq \epsilon, \\ d(\varpi_{i(m)-1}, \varpi_{j(m)}) < \epsilon, \quad \forall m \in \mathbb{N}. \end{aligned}$$

Based on the proof of Theorem 3.1, we have

$$\begin{aligned} \lim_{m \rightarrow \infty} d(\varpi_{i(m)}, \varpi_{j(m)}) &= \lim_{m \rightarrow \infty} d(\varpi_{i(m)-1}, \varpi_{j(m)-1}) \\ &= \epsilon. \end{aligned} \quad (15)$$

From (10), we obtain

$$\begin{aligned} \iota + F(d(\Omega\varpi_{i(m)-1}, \Omega\varpi_{j(m)-1})) \\ \leq F(d(\varpi_{i(m)-1}, \varpi_{j(m)-1})), \quad \forall m \in \mathbb{N}. \end{aligned}$$

This means

$$\begin{aligned} & \iota + F(d(\Omega\varpi_{i(m)}, \Omega\varpi_{j(m)})) \\ & \leq F(d(\varpi_{i(m)-1}, \varpi_{j(m)-1})), \quad \forall m \in \mathbb{N}. \end{aligned}$$

Because $m \rightarrow \infty$, using (F_3'') , we obtain $\iota + F(\epsilon) \leq F(\epsilon)$. Therefore, $\{\varpi_m\}_{m \in \mathbb{N}}$ is an O -Cauchy sequence. Because (Ξ, d) is an orthogonal-complete metric space, $\{\varpi_m\}_{m \in \mathbb{N}}$ converges to a certain point t in Ξ . Next, we prove that t is a fixed-point of Ω . Otherwise, $\Omega t \neq t$. If a sequence $\{i(m)\}_{m \in \mathbb{N}}$ of natural numbers exists such that $\varpi_{i(m)+1} = \Omega\varpi_{i(m)} = \Omega t$, then $\lim_{m \rightarrow \infty} \varpi_{i(m)+1} = t$ and $\Omega t = t$. Therefore, there exists $M \in \mathbb{N}$ such that $\varpi_{m+1} = \Omega\varpi_m \neq \Omega t, \forall m \geq M$. Then, from Eq. (10), we obtain

$$\iota + F(d(\varpi_{m+1}, \Omega t)) \leq F(d(\varpi_m, t)), \quad \forall m \in \mathbb{N}.$$

Because $m \rightarrow \infty$, using (F_3'') , we obtain $\lim_{m \rightarrow \infty} F(d(\varpi_{m+1}, \Omega t)) = -\infty$. Consequently, by (F_2) , we obtain $\lim_{m \rightarrow \infty} d(\varpi_{m+1}, \Omega t) = 0$, which allows $d(t, \Omega t) = 0$. Therefore, t is a fixed point of Ω , and the uniqueness yields the same results as in the proof of Theorem 3.1. $\square$

4. Applications

Recall that, for any $1 \leq p < \infty$, the space $L^p(\Xi, F, \lambda)$ (or $L^p(\Xi)$) consists of all values of complex-valued measurable β on the underlying space Ξ satisfying

$$\int_{\Xi} |\beta(w)|^p d\lambda(w),$$

where F is the σ -algebra of the measurable sets, and λ is the measure. When $p = 1$, the space $L^1(\Xi)$ consists of all integrable functions β on Ξ , and we define the L^1 -norm of β as

$$\|\beta\|_1 = \int_{\Xi} |\beta(w)| d\lambda(w).$$

In this section, using Corollary 3.2, we show the existence of a solution to the following differential equation:

$$\begin{cases} j'(r) = H(r, j(r)), & \text{a.e. } r \in I := [0, T]; \\ j(0) = c, & c \geq 1, \end{cases} \quad (16)$$

where $G : I \times \mathbb{R} \rightarrow \mathbb{R}$ is an integrable function that satisfies the following conditions:

- (i) $G(s, y) \geq 0$ for all $y \geq 0$ and $s \in I$;

- (ii) for each $p, q \in L^1(I)$ with $p(s)q(s) \geq l(s)$ or $p(s)q(s) \geq q(s)$ for all $s \in I$, there exist $\gamma \in L^1(I)$ and $\iota > 0$ such that

$$|G(s, p(s)) - G(s, q(s))| \leq \frac{\gamma(s)}{1 + \iota\gamma(s)} |p(s) - q(s)|, \quad (17)$$

and

$$|p(s) - q(s)| \leq \gamma(s)e^{U(s)}, \quad \forall s \in I,$$

where $U(s) := \int_0^s |\gamma(l)| dl$.

Theorem 4.1. Consider the differential Eq. (16). If (i) and (ii) are satisfied, then the differential Eq. (16) has a unique positive solution.

Proof. Let $\Xi = \{w \in C(I, \mathbb{R}) : w(r) > 0 \forall r \in I\}$. The orthogonality relation $\perp$ on Ξ is defined as

$$p \perp q \iff p(r)q(r) \geq p(r) \text{ or } p(r)q(r) \geq q(r), \quad \forall r \in I.$$

Because $U(r) = \int_0^r |\gamma(s)| ds$, we have $U'(r) = |\gamma(r)|$ for almost all $r \in I$. We define a mapping $d : \Xi \times \Xi \rightarrow [0, \infty)$ as

$$d(p, q) = \|p - q\|_U = \sup_{r \in I} e^{-U(r)} |p(r) - q(r)|, \quad \forall p, q \in \Xi.$$

Thus, (Ξ, d) is a metric space and a complete metric space. We define a mapping $Q : \Xi \rightarrow \Xi$ by

$$(Qp)(r) = c + \int_0^r G(s, p(s)) ds.$$

Now, we show that Q is $\perp$ -preserving. For each $p, q \in W$ with $p \perp q$ and $r \in I$, we have

$$(Qp)(r) = c + \int_0^r G(s, p(s)) ds \geq 1.$$

Accordingly, $[(Qp)(r)][(Qq)(r)] \geq (Qq)(r)$, and thus $(Qp)(r) \perp (Qq)(r)$. Then, Q is $\perp$ -preserving. Let $p, q \in \Xi$, where $p \perp q$. Suppose that $Q(p) \neq Q(q)$. For each $s \in I$, we have

$$|p(s) - q(s)| \leq \gamma(s)e^{U(s)},$$

and thus

$$d(p, q) = \sup_{s \in I} e^{-U(s)} |p(s) - q(s)| \leq \gamma(s).$$

Accordingly,

$$1 + \iota d(p, q) \leq 1 + \iota \gamma(s).$$

From (ii), for each $r \in I$, we derive

$$\begin{aligned} & |(Qp)(r) - (Qq)(r)| \\ & \leq \int_0^r |G(s, p(s)) - G(s, q(s))| ds \\ & \leq \int_0^r \frac{|\gamma(s)|}{1 + \iota \gamma(s)} |p(s) - q(s)| ds \\ & \leq \int_0^r \frac{|\gamma(s)|}{1 + \iota d(p, q)} |p(s) - q(s)| e^{-U(s)} e^{U(s)} ds \\ & \leq \frac{d(p, q)}{1 + \iota d(p, q)} \int_0^r |\gamma(s)| e^{U(s)} ds \\ & \leq \frac{d(p, q)}{1 + \tau d(p, q)} \left(e^{U(r)} - 1 \right), \end{aligned}$$

and thus

$$\begin{aligned} & e^{-U(r)} |(Qp)(r) - (Qq)(r)| \\ & \leq e^{-U(r)} \left(e^{U(r)} - 1 \right) \frac{d(p, q)}{1 + \iota d(p, q)} \\ & = \left(1 - e^{-U(r)} \right) \frac{d(p, q)}{1 + \iota d(p, q)} \\ & \leq \left(1 - e^{-\|\beta\|_1} \right) \frac{d(p, q)}{1 + \iota d(p, q)}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} d(Qp, Qq) & \leq \frac{d(p, q)}{1 + \iota d(p, q)}, \\ \frac{1 + \iota d(p, q)}{d(p, q)} & \leq \frac{1}{d(Qp, Qq)}, \\ \iota + \frac{1}{d(p, q)} & \leq \frac{1}{d(Qp, Qq)}, \\ \tau - \frac{1}{d(Qp, Qq)} & \leq -\frac{1}{d(p, q)}. \end{aligned}$$

Thus, $F: \mathbb{R}^+ \rightarrow \mathbb{R}$ by $F(b) = -\frac{1}{b}$ for all $b > 0$. By Corollary 3.2, Q has a unique fixed point; hence, the differential Eq. (16) has a unique positive solution. $\square$

5. Conclusion

In this article, we demonstrated fixed-point theorems for orthogonal F -contraction mappings on an orthogonal-complete metric space.

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