Convexity-Cum-Concavity on Fuzzy Soft Expert Set with Certain Properties

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Abstract

Molodtsov presented the idea of the soft set theory as a universal scientific apparatus for the provisioning of a parameterization tool. Alkhazaleh and Salleh (2011) characterized the idea of soft expert sets in which the client can understand the assessment of specialists in a single pattern and allow the use of this idea for dynamic issues. In this study, we summarize the idea of a soft expert set to fuzzy soft expert set, which will be progressively viable and helpful. The idea of convex and concave sets is crucial for optimization and related theories. In this investigation, convex and concave fuzzy soft expert sets are characterized first, and a portion of their significant properties are then discussed.

Keywords: Soft set, Fuzzy soft set, Soft expert set, Convex fuzzy soft expert set, Concave fuzzy soft expert set

1. Introduction

Many researchers hope to discover proper answers for mathematical issues that cannot be fathomed using conventional strategies. These issues lie in the way in which customary strategies are unable to deal with the issues of uncertainty in the economy, design, and medications, and in dynamic and types of issues. One of these arrangements is a fuzzy set. In a fuzzy set, a component can be an individual from a set, and somewhat simultaneously, a non-part of an equivalent set. Molodtsov [1] started the idea of soft set theory as a scientific apparatus for managing uncertainties. After Molodtsov's work, a few tasks and the use of soft sets were concentrated by Chen et al. [2] and Maji et al. [3,4]. In addition, Maji et al. [5] presented the concept of a fuzzy soft set, an increasingly broad idea, which is a combination of a fuzzy set and a soft set, and considered its properties, and Roy and Maji [6] utilized this theory to address some dynamic issues. Later, Maji et al. [7] introduced the concept of an intuitionistic fuzzy soft set. Since its appearance, several researchers have presented different methods and algorithms for solving decision-making problems in a soft environment. For example, Feng and Zhou [8] presented a soft discernibility matrix, Cagman and Enginoglu [9] presented the concept of a soft matrix, Hasan et al. [10] discussed a fuzzy model for reducing the risk of insolvent loans in the credit sector as applied in Egypt, and Kim et al. [11] investigated a fuzzy mediation analysis for KOSPI-related variables.

In addition, Arora and Garg [12] presented an algorithm for solving decision-making problems based on aggregation operators under intuitionistic fuzzy soft sets. Perveen et al. [13] also developed the theory of spherical fuzzy soft sets and discussed their applications in decision-making problems.

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Moreover, Garg et al. [14] discussed the characterization of fuzzy number intuitionistic fuzzy soft sets and investigated their properties. Alkhazaleh et al. [15] presented the idea of a possible fuzzy soft set and provided its applications in dynamic and clinical conclusions. Alkhazaleh and Salleh [16] initiated the idea of a soft expert set, in which the client can understand the assessment of all specialists in a single pattern with no activities. After any activity, the client can understand the assessment all things considered. Alkhazaleh and Salleh [17] introduced the concept of a fuzzy soft expert set (FSES) and discussed some operations, such as union, intersection, and complement, and provided examples to explain these operations.

In 2013, Deli [18] characterized the notions of convex and concave sets in a soft set environment and generalized its classical properties. In 2016, Majeed [19] extended the concept and investigated the properties of convex soft sets. She developed notions of a convex hull and cone for soft sets and discussed their generalized properties. In 2018, Salih and Sabir [20] introduced the strictness and strength of convex and concave soft sets. In 2019, Deli [21] reviewed his own concept of soft convexity and extended it to a convex and concave fuzzy soft set. In 2020, Rahman et al. [22] defined convexity-cum-concavity on a hypersoft set (an extension of a soft set, defined by Smarandache [23]). In 2021, Rahman et al. [24,25] translated the classical concept of (m, n) -convexity and m -convexity under fuzzy soft set and soft set environments, respectively. They discussed their classical properties and results with applications in the first and second senses. Some type of inadequacy is observed in these models regarding the consideration of the due status to the opinions of experts.

Convexity is an essential concept in the optimization, recognition, and classification of certain patterns, processing and decomposition of images, antimatroids, discrete event simulation, duality problems, and many other related topics in operation research, mathematical economics, numerical analysis, and other mathematical sciences. Deli [21] provided a mathematical tool to tackle all such problems in a soft set environment. FSES theory is a combination of a fuzzy soft set and expert set. It is a generalized theory because it addresses the limitations of the fuzzy soft set for consideration of expert opinions. To keep the existing literature on convexity in line with such a scenario, the literature demands carving out a conceptual framework for solving such problems under a more generalized version, that is, a FSES. Therefore, in this study, an abstracted and analytical approach is utilized to develop a basic framework of convexity and concavity on FSESs, along with some important

results. Examples of convexity and concavity on FSESs are also presented.

The remaining sections of this paper are arranged as follows. In Section 2, some basic preliminaries are described with examples. Section 3 includes the new definitions of a convex FSES and concave FSES. Moreover, some important results are discussed in detail. In Section 4, some concluding remarks and areas of future direction are provided.

2. Preliminaries

In this section, some basic definitions and terms regarding the main work described in the literature are presented.

Definition 1 ([1]). Let $P(Z)$ denote a power set of Z (universe of discourse), and let F be a collection of parameters defining Z . A pair (B, F) is called a soft set (over Z) if and only if B is a mapping given by

$$B : F \rightarrow P(Z).$$

Definition 2 ([5]). Let I^Z represent the power set of all fuzzy subsets of Z . Let $C \subseteq F$. A pair (G, C) is called a fuzzy soft set, where G is a mapping given by

$$G : C \rightarrow I^Z.$$

Definition 3 ([16]). Assume that Y is a set of specialists (operators), $\vec{1}\vec{7}$ is a collection of conclusions, $T = F \times Y \times \vec{1}\vec{7}$ with $S \subseteq T$, and F is a set of parameters.

A pair (H, S) is known as a *soft expert set* over Z , where H is a mapping given by

$$H : S \rightarrow P(Z).$$

For simplicity, we assume in this paper two-valued opinions only in set $\vec{O}$, that is, $\vec{O} = \{0 = \text{disagree}, 1 = \text{agree}\}$.

Definition 4 ([17]). A pair (H, C) is called a FSES over Z , where H is a mapping given by

$$H : C \rightarrow I^Z,$$

where I^Z the set of all fuzzy subsets of Z .

Definition 5 ([17]). For two FSESs (J, C) and (K, D) over Z , (J, C) is called a fuzzy soft expert subset of (K, D) if

1. $C \subseteq D$.

2. $J(e)$ is a fuzzy subset of $K(e)$, for all $e \in C$.

This is shown as $(J, C) \subseteq (K, D)$. Here, (K, D) is called a fuzzy soft expert superset of (J, C) .

Definition 6 ([17]). The complement of a fuzzy soft expert set (G, P) is denoted by $(G, P)^c$ and is characterized by $(G, P)^c = (G^c, \lceil P)$, where $G^c : S \rightarrow I^Z$ is a mapping given by $G^c(n) = c(S(n)) = 1 - S(n)$ for each $n \in S$ while c is a fuzzy complement.

Definition 7 ([17]). Two fuzzy soft expert sets (J, C) and (K, D) over Z are said to be equal if (J, C) is a fuzzy soft expert subset of (K, D) and (K, D) is a fuzzy soft expert subset of (J, C) .

Definition 8 ([17]). The union of two fuzzy soft expert sets (J, C) and (K, D) over Z indicated by $(J, C) \cup (K, D)$ is the soft expert set (H, L) , where $L = C \cup D$, and $\forall m \in L$,

$$H(m) = \begin{cases} J(m); & m \in C - D, \\ K(m); & m \in D - C, \\ s(J(m), K(m)); & m \in C \cap D, \end{cases}$$

while s is an s -norm.

Definition 9 ([17]). The intersection of two fuzzy soft expert sets (J, C) and (K, D) over Z denoted by $(J, C) \cap (K, D)$ is the fuzzy soft expert set (H, L) , where $L = C \cap D$ and $\forall n \in L$,

$$H(m) = \begin{cases} J(m); & m \in C - D, \\ K(m); & m \in D - C, \\ t(J(m), K(m)); & m \in C \cap D, \end{cases}$$

where t is a t -norm.

3. Convex and Concave Fuzzy Soft Expert Sets

In this section, we describe the definition of a convex FSES and a concave FSES. Here, F denotes the n -dimensional Euclidean space $\mathbb{R}^n$, and Z describes the arbitrary set.

Definition 10. The FSES over F is called a convex FSES if

$$\Gamma_{FSES}(pq_1 + (1 - p)q_2) \supseteq \Gamma_{FSES}(q_1) \cap \Gamma_{FSES}(q_2)$$

for each $q_1, q_2 \in F$, and $p \in \Upsilon^\bullet = [0, 1]$.

Example 3.1. Suppose that an organization manufactures modern brands and intends to proceed with an assessment by certain

specialists regarding these items. Let $Z = \{v_1, v_2, v_3, v_5\}$ be a set of items, $F = \{a_1, a_2, a_3, a_5\}$ be a set of choice parameters, and $S = \{a_1, a_2, a_3\}$ be a subset of F , where

a_1 =simple to utilize,

a_2 =natural,

a_3 =modest, and let $Y = \{s, t, u\}$ be a set of specialists.

Assume that the organization has appropriated a survey to three specialists to settle on choices on the organization's items, and we obtain the following:

$$H_1 = H(b_1, s, 1) = \{v_1, v_2, v_5\},$$

$$H_2 = H(a_1, t, 1) = \{v_3, v_5\},$$

$$H_3 = H(b_1, u, 1) = \{v_3, v_5\},$$

$$H_4 = H(b_2, s, 1) = \{v_4\},$$

$$H_5 = H(a_2, t, 1) = \{v_1, v_3\},$$

$$H_6 = H(b_2, u, 1) = \{v_1, v_2, v_4\},$$

$$H_7 = H(b_3, s, 1) = \{v_3, v_4\},$$

$$H_8 = H(b_3, t, 1) = \{v_1, v_2\},$$

$$H_9 = H(b_3, u, 1) = \{v_4\},$$

$$H_{10} = H(b_1, s, 0) = \{v_3\},$$

$$H_{11} = H(b_1, t, 0) = \{v_2, v_3\},$$

$$H_{12} = H(b_1, u, 0) = \{v_1, v_2\},$$

$$H_{13} = H(b_2, s, 0) = \{v_1, v_2, v_3\},$$

$$H_{14} = H(b_2, t, 0) = \{v_2, v_4\},$$

$$H_{15} = H(b_2, u, 0) = \{v_3\},$$

$$H_{16} = H(b_3, s, 0) = \{v_1, v_2\},$$

$$H_{17} = H(b_3, t, 0) = \{v_3, v_5\},$$

$$H_{18} = H(b_3, u, 0) = \{v_1, v_2, v_3\}.$$

We have the following soft expert set:

$$(H, S) = \left\{ \begin{pmatrix} H_1 = H(b_1, s, 1) = \{v_1, v_2, v_4\}, \\ H_2 = H(b_1, t, 1) = \{v_3, v_4\}, \\ H_3 = H(b_1, u, 1) = \{v_3, v_4\} \end{pmatrix}, \begin{pmatrix} H_4 = H(b_2, s, 1) = \{v_4\}, \\ H_5 = H(b_2, t, 1) = \{v_1, v_3\}, \\ H_6 = H(b_2, u, 1) = \{v_1, v_2, v_4\} \end{pmatrix} \right\}$$

$$\left\{ \begin{array}{l} \left(\begin{array}{l} H_7 = H(b_3, s, 1) = \{v_3, v_4\}, \\ H_8 = H(b_3, t, 1) = \{v_1, v_2\}, \\ H_9 = H(b_3, u, 1) = \{v_4\} \end{array} \right), \\ \left(\begin{array}{l} H_{10} = H(b_1, s, 0) = \{v_3\}, \\ H_{11} = H(b_1, t, 0) = \{v_2, v_3\}, \\ H_{12} = H(b_1, u, 0) = \{v_1, v_2\} \end{array} \right), \\ \left(\begin{array}{l} H_{13} = H(b_2, s, 0) = \{v_1, v_2, v_3\}, \\ H_{14} = H(b_2, t, 0) = \{v_2, v_4\}, \\ H_{15} = H(b_2, u, 0) = \{v_3\} \end{array} \right), \\ \left(\begin{array}{l} H_{16} = H(b_3, s, 0) = \{v_1, v_2\}, \\ H_{17} = H(b_3, t, 0) = \{v_3, v_4\}, \\ H_{18} = H(b_3, u, 0) = \{v_1, v_2, v_3\} \end{array} \right) \end{array} \right\}.$$

Note that from the above pattern, the first specialist, s , “concludes” that the “simple to utilize” items are v_1, v_2 , and v_5 . The subsequent specialist t “concludes” that the “simple to utilize” items are v_1 and v_5 , and the third master u “concludes” that the “simple to utilize” items are v_3 and v_5 . Every specialists “concludes” that item v_5 is “anything but difficult to utilize.”

Example 3.2. Suppose that a company produces new types of products and wants to consider the opinions of some experts about these products. Let $Z = \{w_1, w_2, w_3, w_5\}$ be a set of products, and $E = \{b_1, b_2, b_3\}$ be a set of decision parameters where b_i ($i = 1, 2, 3$) denotes the parameters “easy to use,” “quality,” and “cheap.” Let $X = \{f, g, h\}$ be an expert set. Suppose that

$$\begin{aligned} H_1 &= H(b_1, f, 1) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.7}, \frac{w_3}{0.5}, \frac{w_4}{0.1} \right\}, \\ H_2 &= H(b_1, g, 1) = \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.8}, \frac{w_3}{0.4}, \frac{w_4}{0.2} \right\}, \\ H_3 &= H(b_1, h, 1) = \left\{ \frac{w_1}{0.7}, \frac{w_2}{0.5}, \frac{w_3}{0.6}, \frac{w_4}{0.3} \right\}, \\ H_4 &= H(b_2, f, 1) = \left\{ \frac{w_1}{0.9}, \frac{w_2}{0.4}, \frac{w_3}{0.7}, \frac{w_4}{0.3} \right\}, \\ H_5 &= H(b_2, g, 1) = \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.8}, \frac{w_3}{0.3}, \frac{w_4}{0.2} \right\}, \\ H_6 &= H(b_2, h, 1) = \left\{ \frac{w_1}{0.5}, \frac{w_2}{0.3}, \frac{w_3}{0.6}, \frac{w_4}{0.8} \right\}, \\ H_7 &= H(b_3, f, 1) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.9}, \frac{w_3}{0.4}, \frac{w_4}{0.5} \right\}, \\ H_8 &= H(b_3, g, 1) = \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.6}, \frac{w_3}{0.7}, \frac{w_4}{0.9} \right\}, \\ H_9 &= H(b_3, h, 1) = \left\{ \frac{w_1}{0.7}, \frac{w_2}{0.3}, \frac{w_3}{0.5}, \frac{w_4}{0.2} \right\}, \\ H_{10} &= H(b_1, f, 0) = \left\{ \frac{w_1}{0.3}, \frac{w_2}{0.2}, \frac{w_3}{0.4}, \frac{w_4}{0.1} \right\}, \\ H_{11} &= H(b_1, g, 0) = \left\{ \frac{w_1}{0.1}, \frac{w_2}{0.9}, \frac{w_3}{0.6}, \frac{w_4}{0.2} \right\}, \end{aligned}$$

$$\begin{aligned} H_{12} &= H(b_1, h, 0) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.1}, \frac{w_3}{0.3}, \frac{w_4}{0.5} \right\}, \\ H_{13} &= H(b_2, f, 0) = \left\{ \frac{w_1}{0.8}, \frac{w_2}{0.3}, \frac{w_3}{0.5}, \frac{w_4}{0.7} \right\}, \\ H_{14} &= H(b_2, g, 0) = \left\{ \frac{w_1}{0.7}, \frac{w_2}{0.2}, \frac{w_3}{0.9}, \frac{w_4}{0.4} \right\}, \\ H_{15} &= H(b_2, h, 0) = \left\{ \frac{w_1}{0.6}, \frac{w_2}{0.7}, \frac{w_3}{0.3}, \frac{w_4}{0.2} \right\}, \\ H_{16} &= H(b_3, f, 0) = \left\{ \frac{w_1}{0.1}, \frac{w_2}{0.4}, \frac{w_3}{0.7}, \frac{w_4}{0.8} \right\}, \\ H_{17} &= H(b_3, g, 0) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.9}, \frac{w_3}{0.8}, \frac{w_4}{0.3} \right\}, \\ H_{18} &= H(b_3, h, 0) = \left\{ \frac{w_1}{0.5}, \frac{w_2}{0.3}, \frac{w_3}{0.6}, \frac{w_4}{0.1} \right\}. \end{aligned}$$

The FSES can be described as

$$(H, S) = \left\{ \begin{array}{l} H_1 = H(b_1, f, 1) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.7}, \frac{w_3}{0.5}, \frac{w_4}{0.1} \right\}, \\ H_2 = H(b_1, g, 1) = \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.8}, \frac{w_3}{0.4}, \frac{w_4}{0.2} \right\}, \\ H_3 = H(b_1, h, 1) = \left\{ \frac{w_1}{0.7}, \frac{w_2}{0.5}, \frac{w_3}{0.6}, \frac{w_4}{0.3} \right\}, \\ H_4 = H(b_2, f, 1) = \left\{ \frac{w_1}{0.9}, \frac{w_2}{0.4}, \frac{w_3}{0.7}, \frac{w_4}{0.3} \right\}, \\ H_5 = H(b_2, g, 1) = \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.8}, \frac{w_3}{0.3}, \frac{w_4}{0.2} \right\}, \\ H_6 = H(b_2, h, 1) = \left\{ \frac{w_1}{0.5}, \frac{w_2}{0.3}, \frac{w_3}{0.6}, \frac{w_4}{0.8} \right\}, \\ H_7 = H(b_3, f, 1) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.9}, \frac{w_3}{0.4}, \frac{w_4}{0.5} \right\}, \\ H_8 = H(b_3, g, 1) = \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.6}, \frac{w_3}{0.7}, \frac{w_4}{0.9} \right\}, \\ H_9 = H(b_3, h, 1) = \left\{ \frac{w_1}{0.7}, \frac{w_2}{0.3}, \frac{w_3}{0.5}, \frac{w_4}{0.2} \right\}, \\ H_{10} = H(b_1, f, 0) = \left\{ \frac{w_1}{0.3}, \frac{w_2}{0.2}, \frac{w_3}{0.4}, \frac{w_4}{0.1} \right\}, \\ H_{11} = H(b_1, g, 0) = \left\{ \frac{w_1}{0.1}, \frac{w_2}{0.9}, \frac{w_3}{0.6}, \frac{w_4}{0.2} \right\}, \\ H_{12} = H(b_1, h, 0) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.1}, \frac{w_3}{0.3}, \frac{w_4}{0.5} \right\}, \\ H_{13} = H(b_2, f, 0) = \left\{ \frac{w_1}{0.8}, \frac{w_2}{0.3}, \frac{w_3}{0.5}, \frac{w_4}{0.7} \right\}, \\ H_{14} = H(b_2, g, 0) = \left\{ \frac{w_1}{0.7}, \frac{w_2}{0.2}, \frac{w_3}{0.9}, \frac{w_4}{0.4} \right\}, \\ H_{15} = H(b_2, h, 0) = \left\{ \frac{w_1}{0.6}, \frac{w_2}{0.7}, \frac{w_3}{0.3}, \frac{w_4}{0.2} \right\}, \\ H_{16} = H(b_3, f, 0) = \left\{ \frac{w_1}{0.1}, \frac{w_2}{0.4}, \frac{w_3}{0.7}, \frac{w_4}{0.8} \right\}, \\ H_{17} = H(b_3, g, 0) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.9}, \frac{w_3}{0.8}, \frac{w_4}{0.3} \right\}, \\ H_{18} = H(b_3, h, 0) = \left\{ \frac{w_1}{0.5}, \frac{w_2}{0.3}, \frac{w_3}{0.6}, \frac{w_4}{0.1} \right\} \end{array} \right\}.$$

Example 3.3. Referring to Example 3.2, suppose

$$C = \{H(b_1, f, 1), H(b_2, f, 0), H(b_3, f, 1), H(b_1, g, 1),$$

$$H(b_2, g, 1), H(b_3, g, 0), H(b_1, h, 0), H(b_2, h, 1), \\ H(b_3, h, 1)\},$$

and

$$D = \{H(b_1, f, 1), H(b_2, f, 0), H(b_3, f, 1), H(b_1, g, 1), \\ H(b_2, g, 1), H(b_3, g, 1), H(b_1, h, 0), H(b_2, h, 1)\}.$$

Let (J, C) and (K, D) be two fuzzy soft expert sets, such that

$$(J, C) = \left\{ \begin{array}{l} H_1 = H(b_1, f, 1) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.7}, \frac{w_3}{0.5}, \frac{w_4}{0.1} \right\}, \\ H_2 = H(b_1, g, 1) = \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.8}, \frac{w_3}{0.4}, \frac{w_4}{0.2} \right\}, \\ H_5 = H(b_2, g, 1) = \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.8}, \frac{w_3}{0.3}, \frac{w_4}{0.2} \right\}, \\ H_6 = H(b_2, h, 1) = \left\{ \frac{w_1}{0.5}, \frac{w_2}{0.3}, \frac{w_3}{0.6}, \frac{w_4}{0.8} \right\}, \\ H_7 = H(b_3, f, 1) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.9}, \frac{w_3}{0.4}, \frac{w_4}{0.5} \right\}, \\ H_9 = H(b_3, h, 1) = \left\{ \frac{w_1}{0.7}, \frac{w_2}{0.3}, \frac{w_3}{0.5}, \frac{w_4}{0.2} \right\}, \\ H_{12} = H(b_1, h, 0) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.1}, \frac{w_3}{0.3}, \frac{w_4}{0.5} \right\}, \\ H_{13} = H(b_2, f, 0) = \left\{ \frac{w_1}{0.8}, \frac{w_2}{0.3}, \frac{w_3}{0.5}, \frac{w_4}{0.7} \right\}, \\ H_{17} = H(b_3, g, 0) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.9}, \frac{w_3}{0.8}, \frac{w_4}{0.3} \right\} \end{array} \right\},$$

$$(K, D) = \left\{ \begin{array}{l} H_1 = H(b_1, f, 1) = \left\{ \frac{w_1}{0.1}, \frac{w_2}{0.8}, \frac{w_3}{0.4}, \frac{w_4}{0.2} \right\}, \\ H_2 = H(b_1, g, 1) = \left\{ \frac{w_1}{0.5}, \frac{w_2}{0.7}, \frac{w_3}{0.2}, \frac{w_4}{0.3} \right\}, \\ H_5 = H(b_2, g, 1) = \left\{ \frac{w_1}{0.6}, \frac{w_2}{0.3}, \frac{w_3}{0.2}, \frac{w_4}{0.4} \right\}, \\ H_6 = H(b_2, h, 1) = \left\{ \frac{w_1}{0.3}, \frac{w_2}{0.4}, \frac{w_3}{0.7}, \frac{w_4}{0.9} \right\}, \\ H_7 = H(b_3, f, 1) = \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.1}, \frac{w_3}{0.5}, \frac{w_4}{0.6} \right\}, \\ H_9 = H(b_3, h, 1) = \left\{ \frac{w_1}{0.9}, \frac{w_2}{0.2}, \frac{w_3}{0.4}, \frac{w_4}{0.3} \right\}, \\ H_{12} = H(b_1, h, 0) = \left\{ \frac{w_1}{0.3}, \frac{w_2}{0.2}, \frac{w_3}{0.4}, \frac{w_4}{0.6} \right\}, \\ H_{13} = H(b_2, f, 0) = \left\{ \frac{w_1}{0.7}, \frac{w_2}{0.2}, \frac{w_3}{0.4}, \frac{w_4}{0.5} \right\}, \\ H_8 = H(b_3, g, 1) = \left\{ \frac{w_1}{0.3}, \frac{w_2}{0.4}, \frac{w_3}{0.6}, \frac{w_4}{0.8} \right\} \end{array} \right\}.$$

By applying the union operation(max), we obtain

$$(H, L) = \left\{ \begin{array}{l} H_1 = H(b_1, f, 1) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.8}, \frac{w_3}{0.5}, \frac{w_4}{0.2} \right\}, \\ H_2 = H(b_1, g, 1) = \left\{ \frac{w_1}{0.5}, \frac{w_2}{0.8}, \frac{w_3}{0.4}, \frac{w_4}{0.3} \right\}, \\ H_5 = H(b_2, g, 1) = \left\{ \frac{w_1}{0.6}, \frac{w_2}{0.8}, \frac{w_3}{0.4}, \frac{w_4}{0.4} \right\}, \\ H_6 = H(b_2, h, 1) = \left\{ \frac{w_1}{0.5}, \frac{w_2}{0.9}, \frac{w_3}{0.7}, \frac{w_4}{0.9} \right\}, \end{array} \right\}$$

$$\left\{ \begin{array}{l} H_7 = H(b_3, f, 1) = \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.9}, \frac{w_3}{0.5}, \frac{w_4}{0.6} \right\}, \\ H_9 = H(b_3, h, 1) = \left\{ \frac{w_1}{0.9}, \frac{w_2}{0.3}, \frac{w_3}{0.5}, \frac{w_4}{0.3} \right\}, \\ H_{12} = H(b_1, h, 0) = \left\{ \frac{w_1}{0.3}, \frac{w_2}{0.2}, \frac{w_3}{0.4}, \frac{w_4}{0.6} \right\}, \\ H_{13} = H(b_2, f, 0) = \left\{ \frac{w_1}{0.8}, \frac{w_2}{0.3}, \frac{w_3}{0.5}, \frac{w_4}{0.7} \right\}, \\ H_8 = H(b_3, g, 1) = \left\{ \frac{w_1}{0.3}, \frac{w_2}{0.4}, \frac{w_3}{0.6}, \frac{w_4}{0.8} \right\}, \\ H_{17} = H(b_3, g, 0) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.9}, \frac{w_3}{0.8}, \frac{w_4}{0.3} \right\} \end{array} \right\}.$$

Example 3.4. Referring to Example 3.2, let

$$C = \{H(b_1, f, 1), H(b_2, f, 0), H(b_3, f, 1), \\ H(b_1, g, 1), H(b_2, g, 1), H(b_3, g, 0), \\ H(b_1, h, 0), H(b_2, h, 1), H(b_3, h, 1)\},$$

and

$$D = \{H(b_1, f, 1), H(b_2, f, 0), H(b_3, f, 1), \\ H(b_1, g, 1), H(b_2, g, 1), H(b_3, g, 1), \\ H(b_1, h, 0), H(b_2, h, 1)\}.$$

Let (J, C) and (K, D) be two FSESs, such that

$$(J, C) = \left\{ \begin{array}{l} H_1 = H(b_1, f, 1) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.7}, \frac{w_3}{0.5}, \frac{w_4}{0.1} \right\}, \\ H_2 = H(b_1, g, 1) = \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.8}, \frac{w_3}{0.4}, \frac{w_4}{0.2} \right\}, \\ H_5 = H(b_2, g, 1) = \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.8}, \frac{w_3}{0.3}, \frac{w_4}{0.2} \right\}, \\ H_6 = H(b_2, h, 1) = \left\{ \frac{w_1}{0.5}, \frac{w_2}{0.3}, \frac{w_3}{0.6}, \frac{w_4}{0.8} \right\}, \\ H_7 = H(b_3, f, 1) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.9}, \frac{w_3}{0.4}, \frac{w_4}{0.5} \right\}, \\ H_9 = H(b_3, h, 1) = \left\{ \frac{w_1}{0.7}, \frac{w_2}{0.3}, \frac{w_3}{0.5}, \frac{w_4}{0.2} \right\}, \\ H_{12} = H(b_1, h, 0) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.1}, \frac{w_3}{0.3}, \frac{w_4}{0.5} \right\}, \\ H_{13} = H(b_2, f, 0) = \left\{ \frac{w_1}{0.8}, \frac{w_2}{0.3}, \frac{w_3}{0.5}, \frac{w_4}{0.7} \right\}, \\ H_{17} = H(b_3, g, 0) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.9}, \frac{w_3}{0.8}, \frac{w_4}{0.3} \right\} \end{array} \right\},$$

$$(K, D) = \left\{ \begin{array}{l} H_1 = H(b_1, f, 1) = \left\{ \frac{w_1}{0.1}, \frac{w_2}{0.8}, \frac{w_3}{0.4}, \frac{w_4}{0.2} \right\}, \\ H_2 = H(b_1, g, 1) = \left\{ \frac{w_1}{0.5}, \frac{w_2}{0.7}, \frac{w_3}{0.2}, \frac{w_4}{0.3} \right\}, \\ H_5 = H(b_2, g, 1) = \left\{ \frac{w_1}{0.6}, \frac{w_2}{0.3}, \frac{w_3}{0.2}, \frac{w_4}{0.4} \right\}, \\ H_6 = H(b_2, h, 1) = \left\{ \frac{w_1}{0.3}, \frac{w_2}{0.4}, \frac{w_3}{0.7}, \frac{w_4}{0.9} \right\}, \end{array} \right\}$$

$$\left\{ \begin{array}{l} H_7 = H(b_3, f, 1) = \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.1}, \frac{w_3}{0.5}, \frac{w_4}{0.6} \right\}, \\ H_9 = H(b_3, h, 1) = \left\{ \frac{w_1}{0.9}, \frac{w_2}{0.2}, \frac{w_3}{0.4}, \frac{w_4}{0.3} \right\}, \\ H_{12} = H(b_1, h, 0) = \left\{ \frac{w_1}{0.3}, \frac{w_2}{0.2}, \frac{w_3}{0.4}, \frac{w_4}{0.6} \right\}, \\ H_{13} = H(b_2, f, 0) = \left\{ \frac{w_1}{0.7}, \frac{w_2}{0.2}, \frac{w_3}{0.4}, \frac{w_4}{0.5} \right\}, \\ H_8 = H(b_3, g, 1) = \left\{ \frac{w_1}{0.3}, \frac{w_2}{0.4}, \frac{w_3}{0.6}, \frac{w_4}{0.8} \right\} \end{array} \right\}.$$

By applying the intersection operation(min), we obtain

$$(K, L) = \left\{ \begin{array}{l} H_1 = H(b_1, f, 1) = \left\{ \frac{w_1}{0.1}, \frac{w_2}{0.7}, \frac{w_3}{0.4}, \frac{w_4}{0.1} \right\}, \\ H_2 = H(b_1, g, 1) = \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.7}, \frac{w_3}{0.2}, \frac{w_4}{0.2} \right\}, \\ H_5 = H(b_2, g, 1) = \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.3}, \frac{w_3}{0.2}, \frac{w_4}{0.2} \right\}, \\ H_6 = H(b_2, h, 1) = \left\{ \frac{w_1}{0.3}, \frac{w_2}{0.3}, \frac{w_3}{0.6}, \frac{w_4}{0.8} \right\}, \\ H_7 = H(b_3, f, 1) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.1}, \frac{w_3}{0.4}, \frac{w_4}{0.5} \right\}, \\ H_9 = H(b_3, h, 1) = \left\{ \frac{w_1}{0.7}, \frac{w_2}{0.2}, \frac{w_3}{0.4}, \frac{w_4}{0.2} \right\}, \\ H_{12} = H(b_1, h, 0) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.1}, \frac{w_3}{0.3}, \frac{w_4}{0.5} \right\}, \\ H_{13} = H(b_2, f, 0) = \left\{ \frac{w_1}{0.7}, \frac{w_2}{0.2}, \frac{w_3}{0.4}, \frac{w_4}{0.5} \right\} \end{array} \right\}.$$

Example 3.5. Referring to Example 3.2 with $S = \{b_1, b_2, b_3\} = \{1, 2, 3\}$, and $Y = \{f, g, h\} = \{1, 2, 3\}$, $H_1 = \Gamma_{FSES}(1, 1, 1) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.7}, \frac{w_3}{0.5}, \frac{w_4}{0.1} \right\}$, $H_2 = \Gamma_{FSES}(1, 2, 1) = \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.8}, \frac{w_3}{0.4}, \frac{w_4}{0.2} \right\}$.

Take $n = 0.6$, and $q_1 = (1, 1, 1)$ and $q_2 = (1, 2, 1)$, then $\Gamma_{FSES}(pq_1 + (1 - p)q_2) = \Gamma_{FSES}(0.6(1, 1, 1) + (1 - 0.6)(1, 2, 1)) = \Gamma_{FSES}((0.6, 0.6, 0.6) + (0.4, 0.8, 0.4)) = \Gamma_{FSES}(1, 1.4, 1) = \Gamma_{FSES}(1, 1, 1) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.7}, \frac{w_3}{0.5}, \frac{w_4}{0.1} \right\}$, and $\Gamma_{FSES}(q_1) \cap \Gamma_{FSES}(q_2) = \Gamma_{FSES}(1, 1, 1) \cap \Gamma_{FSES}(1, 2, 1) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.7}, \frac{w_3}{0.5}, \frac{w_4}{0.1} \right\} \cap \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.8}, \frac{w_3}{0.4}, \frac{w_4}{0.2} \right\} = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.7}, \frac{w_3}{0.4}, \frac{w_4}{0.1} \right\}$.

Clearly,

$$\Gamma_{FSES}(pq_1 + (1 - p)q_2) \supseteq \Gamma_{FSES}(q_1) \cap \Gamma_{FSES}(q_2).$$

Definition 11. The FSES on F is said to be a concave FSES if

$$\Gamma_{FSES}(pq_1 + (1 - p)q_2) \subseteq \Gamma_{FSES}(q_1) \cup \Gamma_{FSES}(q_2),$$

for each $q_1, q_2 \in F$ and $p \in \Upsilon^\bullet$.

Example 3.6. Considering the data from Example 3.2, we have

$$\Gamma_{FSES}(q_1) \cup \Gamma_{FSES}(q_2)$$

$$\begin{aligned} &= \Gamma_{FSES}(1, 1, 1) \cup \Gamma_{FSES}(1, 2, 1) \\ &= \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.7}, \frac{w_3}{0.5}, \frac{w_4}{0.1} \right\} \cup \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.8}, \frac{w_3}{0.4}, \frac{w_4}{0.2} \right\} \\ &= \left\{ \frac{w_1}{0.4}, \frac{w_2}{0.8}, \frac{w_3}{0.5}, \frac{w_4}{0.2} \right\}, \end{aligned}$$

and $\Gamma_{FSES}(pq_1 + (1 - p)q_2) = \left\{ \frac{w_1}{0.2}, \frac{w_2}{0.7}, \frac{w_3}{0.5}, \frac{w_4}{0.1} \right\}$, respectively. Clearly,

$$\Gamma_{FSES}(pq_1 + (1 - p)q_2) \subseteq \Gamma_{FSES}(q_1) \cup \Gamma_{FSES}(q_2).$$

Definition 12. Suppose R is a FSES over Z , and ρ is a subset of Z . Subsequently, the ρ -inclusion of R is defined as

$$R^\rho = \{\epsilon \in F : g_R(\epsilon) \supseteq \rho\}.$$

Theorem 3.7. $H_1 \cap H_2$ is a convex fuzzy soft expert set, whereas H_1 and H_2 are convex FSESs.

Proof. Assume $\exists q_1, q_2 \in F$ and $p \in [0, 1]$, and $H_3 = H_1 \cap H_2$.

$$\begin{aligned} &\Gamma_{FSES(H_3)}(pq_1 + (1 - p)q_2) \\ &= \Gamma_{FSES(H_1)}(pq_1 + (1 - p)q_2) \cap \Gamma_{FSES(H_2)}(pq_1 + (1 - p)q_2). \end{aligned}$$

$\therefore H_1$ and H_2 are convex FSESs, and thus

$$\begin{aligned} &\Gamma_{FSES(H_1)}(pq_1 + (1 - p)q_2) \\ &\supseteq \Gamma_{FSES(H_1)}(q_1) \cap \Gamma_{FSES(H_1)}(q_2), \\ &\Gamma_{FSES(H_2)}(pq_1 + (1 - p)q_2) \\ &\supseteq \Gamma_{FSES(H_2)}(q_1) \cap \Gamma_{FSES(H_2)}(q_2). \\ \therefore &\Gamma_{FSES(H_3)}(pq_1 + (1 - p)q_2) \\ &\supseteq (\Gamma_{FSES(H_1)}(q_1) \cap \Gamma_{FSES(H_1)}(q_2)) \\ &\quad \cap (\Gamma_{FSES(H_2)}(q_1) \cap \Gamma_{FSES(H_2)}(q_2)), \\ &\Gamma_{FSES(H_3)}(pq_1 + (1 - p)q_2) \\ &\supseteq \Gamma_{FSES(H_3)}(q_1) \cap \Gamma_{FSES(H_3)}(q_2), \end{aligned}$$

and hence the theorem is proved. $\square$

Note that with this theorem, the intersection of two convex FSESs is proved to be a convex FSES; this can be generalized for more than two convex FSESs. Hence, we can generalize this result to a countable number of convex FSESs.

Remark 3.8. If $\{8_n : n \in \{1, 2, 3, \dots\}\}$ is any countable collection of convex FSESs, then $\bigcap_{n \in I} 8_n$ is a convex FSES.

Remark 3.9. If $\{8_n : n \in \{1, 2, 3, \dots\}\}$ is any countable collection of convex FSESs, then $\bigcup_{n \in I} 8_n$ is not a convex FSES.

Theorem 3.10. Here, G is a convex FSES on F iff for each

$\eta \in [0, 1]$ and $\rho \in P(Z)$, and G^ρ is a convex FSES on F .

Proof. Suppose that G is convex FSES. If $q_1, q_2 \in F$ and $\rho \in P(Z)$, then $\Gamma_{FSES(G)}(q_1) \supseteq \rho$ and $\Gamma_{FSES(G)}(q_2) \supseteq \rho$. Based on the convexity of G , we have

$$\begin{aligned} & \Gamma_{FSES(G)}(\eta q_1 + (1 - \eta) q_2) \\ & \supseteq \Gamma_{FSES(G)}(q_1) \cap \Gamma_{FSES(G)}(q_2), \end{aligned}$$

and thus G^ρ is convex FSES.

Conversely, suppose that G^ρ is a convex FSES for each $\eta \in [0, 1]$. Then, for $q_1, q_2 \in F$, G^ρ is convex for $\rho = \Gamma_{FSES(G)}(q_1) \cap \Gamma_{FSES(G)}(q_2)$.

$\therefore \Gamma_{FSES(G)}(q_1) \supseteq \rho$ and $\Gamma_{FSES(G)}(q_2) \supseteq \rho$, we have $q_1 \in G^\rho$ and $q_2 \in G^\rho$,

hence $\eta q_1 + (1 - \eta) q_2 \in G^\rho$.

$$\begin{aligned} \therefore & \Gamma_{FSES(G)}(\eta q_1 + (1 - \eta) q_2) \\ & \supseteq \Gamma_{FSES(G)}(q_1) \cap \Gamma_{FSES(G)}(q_2), \end{aligned}$$

which implies that G is convex FSES. $\square$

Note here that the inclusive property of a convex FSES is discussed. This may extend to other operations such as a union, intersection, AND, and OR of FSESs.

Theorem 3.11. $G_1 \cup G_2$ is concave FSES, whereas the two sets G_1 and G_2 are concave FSESs.

Proof. Assume that $\exists q_1, q_2 \in F$ and $n \in \Upsilon^\bullet$ and $G_3 = G_1 \cup G_2$. Therefore,

$$\begin{aligned} & \Gamma_{FSES(G_3)}(pq_1 + (1 - p) q_2) \\ & = \Gamma_{FSES(G_1)}(pq_1 + (1 - p) q_2) \\ & \quad \cup \Gamma_{FSES(G_2)}(pq_1 + (1 - p) q_2). \end{aligned}$$

$\therefore G_1$ and G_2 are concave FSESs,

$$\begin{aligned} & \Gamma_{FSES(G_1)}(pq_1 + (1 - p) q_2) \\ & \subseteq \Gamma_{FSES(G_1)}(q_1) \cup \Gamma_{FSES(G_1)}(q_2), \\ & \Gamma_{FSES(G_2)}(pq_1 + (1 - p) q_2) \\ & \subseteq \Gamma_{FSES(G_2)}(q_1) \cup \Gamma_{FSES(G_2)}(q_2). \\ \therefore & \Gamma_{FSES(G_3)}(pq_1 + (1 - p) q_2) \\ & \subseteq (\Gamma_{FSES(G_1)}(q_1) \cup \Gamma_{FSES(G_1)}(q_2)) \\ & \quad \cup (\Gamma_{FSES(G_2)}(q_1) \cup \Gamma_{FSES(G_2)}(q_2)), \end{aligned}$$

and hence

$$\Gamma_{FSES(G_3)}(pq_1 + (1 - p) q_2)$$

$$\subseteq \Gamma_{FSES(G_3)}(q_1) \cup \Gamma_{FSES(G_3)}(q_2).$$

Thus, the theorem is proved. $\square$

Note that we have proved herein that the union of two concave FSESs is again a concave FSES, which can be proved for more than two concave FSESs. Hence, we can generalize this result to a countable number of concave FSESs.

Remark 3.12. If $\{7_n : n \in \{1, 2, 3, \dots\}\}$ is any collection of concave FSESs, then $\bigcup_{n \in I} 7_n$ is a concave FSES.

Remark 3.13. If $\{12_n : n \in \{1, 2, 3, \dots\}\}$ is any collection of convex FSESs, then $\bigcap_{n \in I} 12_n$ is a convex FSES.

Theorem 3.14. Here, K' is concave FSES, whereas K is convex FSES.

Proof. Assume that $\exists q_1, q_2 \in F, p \in \Upsilon^\bullet$.

$\therefore K$ is a convex FSES,

$$\begin{aligned} & \Gamma_{FSES(K)}(pq_1 + (1 - p) q_2) \\ & \supseteq \Gamma_{FSES(K)}(q_1) \cap \Gamma_{FSES(K)}(q_2), \end{aligned}$$

or

$$\begin{aligned} & Z \setminus \Gamma_{FSES(K)}(pq_1 + (1 - p) q_2) \\ & \subseteq Z \setminus \{\Gamma_{FSES(K)}(q_1) \cap \Gamma_{FSES(K)}(q_2)\}. \end{aligned}$$

If $\Gamma_{FSES(K)}(q_1) \supseteq \Gamma_{FSES(K)}(q_2)$, then we may write

$$Z \setminus \Gamma_{FSES(K)}(pq_1 + (1 - p) q_2) \subseteq Z \setminus \Gamma_{FSES(K)}(q_2). \quad (1)$$

If $\Gamma_{FSES(K)}(q_1) \subseteq \Gamma_{FSES(K)}(q_2)$, then we may write

$$Z \setminus \Gamma_{FSES(K)}(pq_1 + (1 - p) q_2) \subseteq Z \setminus \Gamma_{FSES(K)}(q_1). \quad (2)$$

From Eqs. (1) and (2), we have

$$\begin{aligned} & Z \setminus \Gamma_{FSES(K)}(pq_1 + (1 - p) q_2) \\ & \subseteq \{Z \setminus \Gamma_{FSES(K)}(q_1) \cup Z \setminus \Gamma_{FSES(K)}(q_2)\}. \end{aligned} \quad (3)$$

Thus, K' is concave FSES. $\square$

Note that this result can be extended to the complement of the union and intersection of two or more FSESs. Hence, we can generalize this result to De Morgan's law. The converse of this theorem is also true, that is, if one set is concave, then its complement is convex.

Theorem 3.15. Here, L' is convex FSES, whereas L is a concave FSES.

Proof. Assume that $\exists q_1, q_2 \in F, p \in \Upsilon^\bullet$.

$\therefore L$ is a concave FSES,

$$\begin{aligned} & \Gamma_{FSES(L)}(pq_1 + (1-p)q_2) \\ & \subseteq \Gamma_{FSES(L)}(q_1) \cup \Gamma_{FSES(L)}(q_2), \end{aligned}$$

or

$$\begin{aligned} & Z \setminus \Gamma_{FSES(L)}(pq_1 + (1-p)q_2) \\ & \supseteq Z \setminus \{ \Gamma_{FSES(L)}(q_1) \cup \Gamma_{FSES(L)}(q_2) \}. \end{aligned}$$

If $\Gamma_{FSES(L)}(q_1) \supseteq \Gamma_{FSES(L)}(q_2)$ we may write

$$Z \setminus \Gamma_{FSES(L)}(pq_1 + (1-p)q_2) \supseteq Z \setminus \Gamma_{FSES(L)}(q_1). \quad (4)$$

If $\Gamma_{FSES(L)}(q_1) \subseteq \Gamma_{FSES(L)}(q_2)$; therefore

$$Z \setminus \Gamma_{FSES(L)}(pq_1 + (1-p)q_2) \supseteq Z \setminus \Gamma_{FSES(L)}(q_2). \quad (5)$$

From (5) and (5), we obtain

$$\begin{aligned} & Z \setminus \Gamma_{FSES(L)}(pq_1 + (1-p)q_2) \\ & \supseteq \{ Z \setminus \Gamma_{FSES(L)}(q_1) \cap Z \setminus \Gamma_{FSES(L)}(q_2) \}, \end{aligned} \quad (6)$$

which shows that L' is convex FSES. $\square$

Note that this is the opposite of the theorem above. Similarly, this result can be extended to two or more sets. Hence, this result can be generalized to De Morgan's law.

Theorem 3.16. Here, M is a concave FSES on F iff for each $\eta \in [0, 1]$ and $\rho \in P(Z)$, M^ρ is a concave FSES on F .

Proof. Suppose that M is a concave FSES. If $q_1, q_2 \in F$ and $\rho \in P(Z)$, then $\Gamma_{FSES(M)}(q_1) \supseteq \rho$ and $\Gamma_{FSES(M)}(q_2) \supseteq \rho$. By the concavity of M , we have

$$\begin{aligned} & \Gamma_{FSES(M)}(\eta q_1 + (1-\eta)q_2) \\ & \subseteq \Gamma_{FSES(M)}(q_1) \cup \Gamma_{FSES(M)}(q_2), \end{aligned}$$

and hence M^ρ is a concave FSES.

Conversely, suppose that M^ρ is a concave FSES for each $\eta \in [0, 1]$. Subsequently, for $q_1, q_2 \in F$, M^ρ is concave for $\rho = \Gamma_{FSES(M)}(q_1) \cup \Gamma_{FSES(M)}(q_2)$. Therefore, $\Gamma_{FSES(M)}(q_1) \supseteq \rho$ and $\Gamma_{FSES(M)}(q_2) \supseteq \rho$, we have $q_1 \in M^\rho$ and $q_2 \in M^\rho$. Hence, $\eta q_1 + (1-\eta)q_2 \in M^\rho$.

$$\begin{aligned} & \Gamma_{FSES(M)}(\eta q_1 + (1-\eta)q_2) \\ & \subseteq \Gamma_{FSES(M)}(q_1) \cup \Gamma_{FSES(M)}(q_2), \end{aligned}$$

and it is clear that M is a concave FSES. $\square$

Note here that the inclusive property of a concave FSES is discussed. This property can be discussed with other operations such as the union and intersection of FSESs.

4. Conclusion

In this study, convexity-cum-concavity was developed for a FSES. Some of its essential properties, that is, its complement, intersection, and union, have been proved. Future studies may include the following:

- The introduction of strictly and strongly convexity-cum-concavity on a fuzzy soft expert.
- The characterization of a convex hull and convex cone on a fuzzy soft expert.
- The development of many other variants of convexity such as (m, n) -convexity and (m, n) -convexity in the first sense and second senses, e.g., ϕ -convexity, graded convexity, triangular convexity, harmonic convexity, and concave convexity on a FSES.

It may also include an extension of this study by considering the modified versions of the complement, intersection, and union.

Conflict of Interest

No potential conflicts of interest relevant to this article are reported.

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