Strong Form of Soft Semi-Open Sets in Soft Topological Spaces

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Abstract

Soft ω_s -open sets as a class of soft sets that lies strictly between soft open sets and soft semi-open sets is introduced. The natural properties of soft ω_s -open sets are described. Using soft ω_s -open sets, soft ω_s -closure and soft ω_s -interior as new soft operators are defined and investigated. Furthermore, the relationships regarding generated soft topological spaces and generated topological spaces are studied.

Keywords: Soft ω -open sets, Soft semi-open sets, Soft ω_s -open, Generated soft topology

1. Introduction

Throughout this paper, we follow the notions and terminologies as used in [1] and [2], and for simplicity, STS stands for soft topological space. For the purpose of dealing with uncertain objects, Molodtsov [3] introduced soft sets in 1999. Let Y be a universal set and B be a set of parameters. A soft set over Y relative to B is a function $H : B \rightarrow \mathcal{P}(Y)$. The family of all soft sets over Y relative to B is denoted by $SS(Y, B)$. Let $H \in SS(Y, B)$, then H is a countable soft set if $H(b)$ is a countable set for all $b \in B$. The family of all members of $SS(Y, B)$ that are countable is denoted by $CSS(Y, B)$. Furthermore, the null soft set and the absolute soft set are denoted by 0_B and 1_B , respectively. The authors in [4] define the notion of STSs as follows: The triplet (Y, τ, B) , where $\tau \subseteq SS(Y, B)$ is a STS if 0_B and $1_B \in \tau$, τ is closed under finite soft intersection, and τ is closed under arbitrary soft union. If (Y, τ, B) is a STS, then the members of τ are called soft open sets and their soft complements are called soft closed sets. Topologists recently have applied various topological concepts to soft topological spaces ([1,2,5–25]).

Chen [26] introduced the concept of soft semi-open sets, a weaker form of soft open sets in STSs. This paper was followed by many more papers about soft semi-open sets and their modifications. In this paper, we introduce soft ω_s -open sets as a class of soft sets that lies strictly between soft open sets and soft semi-open sets. We present the natural properties of soft ω_s -open sets. Furthermore, using ω_s -open sets, we define and investigate soft ω_s -closure and ω_s -interior as new soft topological operators. We also compare our proposed concepts and their corresponding previous analogous concepts in Theorems 2.2, 2.17, 3.3, and 3.19, with the comparisons supported by examples. We present several sufficient conditions for the equivalence between the proposed concepts and their corresponding previous analogous concepts. We next discuss relationships between relative topological spaces and generated soft topological spaces. Finally, we raise an open question related to soft ω_s -open.

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In [27,28], the authors showed that soft sets are a class of special information systems. This insight constitutes our motivation to study the structures of soft sets for information systems. Therefore, this paper does not only form the theoretical basis for further applications of soft topology, such as soft ω_s -mappings and soft ω_s -compactness, but also comments on the future development of information systems.

The following definitions and results will be used in the sequel:

Definition 1.1. Let $(X, \mathfrak{S})$ be a topological space and let $D \subseteq X$. The D is called

(a) [29] semi-open if there is $V \in \mathfrak{S}$ such that $V \subseteq D \subseteq Cl_{\mathfrak{S}}(V)$. $SO(X, \mathfrak{S})$ will denote the family of all semi-open sets in $(X, \mathfrak{S})$.

(b) [30] ω_s -open if there is $V \in \mathfrak{S}$ such that $V \subseteq D \subseteq Cl_{\mathfrak{S}_\omega}(V)$. $\omega_s(X, \mathfrak{S})$ will denote the family of all ω_s -open sets in $(X, \mathfrak{S})$.

Definition 1.2 [26]. Let (X, τ, A) be a STS and let $K \in SS(X, A)$. Then

(a) K is called a soft semi-open set in (X, τ, A) if there exists $F \in \tau$ such that $F \subseteq K \subseteq Cl_\tau(F)$. $SO(X, \tau, A)$ will denote the family of all soft semi-open sets in (X, τ, A) .

(b) K is called a soft semi-closed set in (X, τ, A) if $1_A - K \in SO(X, \tau, A)$.

Definition 1.3 [26]. Let (X, τ, A) be a STS and $H \in SS(X, A)$.

(a) The soft semi-closure of H in (X, τ, A) is denoted by $S-Cl_\tau(H)$ and defined by $S-Cl_\tau(H) = \bigcap \{M : M \text{ is soft semi-closed in } (X, \tau, A) \text{ and } H \subseteq M\}$.

(b) The soft semi-interior of H in (X, τ, A) is denoted by $S-int_\tau(H)$ and defined by $S-int_\tau(H) = \bigcup \{K : K \text{ is soft semi-open in } (X, \tau, A) \text{ and } K \subseteq H\}$.

As defined in [2], a STS (X, τ, A) is called soft anti-locally countable if for every $F \in \tau - \{0_A\}$, $F \notin CSS(X, A)$.

Proposition 1.4 [2]. Let (X, τ, A) be soft anti-locally countable. Then for all $G \in \tau_\omega$, $Cl_\tau(G) = Cl_{\tau_\omega}(G)$. As defined in [2], a STS (X, τ, A) is called soft locally countable if for $a_x \in SP(X, A)$ there exists $G \in CSS(X, A) \cap \tau$ such that $a_x \tilde{\in} G$.

Proposition 1.5 [2]. A STS (X, τ, A) is soft locally countable if and only if (X, τ_ω, A) is a discrete STS.

Proposition 1.6 [2]. For any STS (X, τ, A) we have $\tau_\omega = (\tau_\omega)_\omega$.

Proposition 1.7 [26]. Let (X, τ, A) be a STS and $H \in SS(X, A)$. Then

(a) $S-Cl_\tau(H)$ is the smallest soft semi-closed in (X, τ, A) containing H .

(b) H is soft semi-closed in (X, τ, A) if and only if $H = S-Cl_\tau(H)$.

2. Soft ω_s -Open sets

Definition 2.1. Let (X, τ, A) be a STS and let $K \in SS(X, A)$. Then K is called a soft ω_s -open set in (X, τ, A) if there exists $F \in \tau$ such that $F \subseteq K \subseteq Cl_{\tau_\omega}(F)$. $\omega_s(X, \tau, A)$ will denote the family of all soft ω_s -open sets in (X, τ, A) .

Theorem 2.2. Let (X, τ, A) be a STS. Then $\tau \subseteq \omega_s(X, \tau, A) \subseteq SO(X, \tau, A)$.

Proof. Let $G \in \tau$, choose $F = G$. Then $F \in \tau$ and $F \subseteq G \subseteq Cl_{\tau_\omega}(F)$ which implies that $G \in \omega_s(X, \tau, A)$. This shows that $\tau \subseteq \omega_s(X, \tau, A)$. Let $K \in \omega_s(X, \tau, A)$, then by definition we find $F \in \tau$ such that $F \subseteq K \subseteq Cl_{\tau_\omega}(F) \subseteq Cl_\tau(F)$ which implies that $K \in SO(X, \tau, A)$. It follows that $\omega_s(X, \tau, A) \subseteq SO(X, \tau, A)$. $\square$

The following three examples show that none of the two inclusions in Theorem 2.2 is equality in general:

Example 2.3. Let $X = \mathbb{R}$ and $A = [0, 1]$. Let $F, G \in SS(X, A)$ such that for every $a \in A$, $F(a) = \mathbb{N}$ and $G(a) = \mathbb{Q}^c$. Let $\tau = \{0_A, 1_A, F, G, F \tilde{\cup} G\}$. One can easily check that $Cl_{\tau_\omega}(F) = F$, $Cl_\tau(F) = 1_A - G$ and $Cl_{\tau_\omega}(G) = 1_A - F$. Hence $1_A - G \in SO(X, \tau, A) - \omega_s(X, \tau, A)$ and $1_A - F \in \omega_s(X, \tau, A) - \tau$.

Example 2.4. Let $X = \mathbb{R}$ and $A = \mathbb{N}$. Let $\tau = \{F \in SS(X, A) : F(a) \text{ belongs to the usual topology on } \mathbb{R}\}$.

Let $G \in SS(X, A)$ such that for every $a \in A$, $G(a) = [0, 1]$. Then $G \in \omega_s(X, \tau, A) - \tau$.

Example 2.5. Let $X = \mathbb{Q}$ and $A = \mathbb{R}$. Let $F, G \in SS(X, A)$ such that for every $a \in A$, $F(a) = \{1\}$ and $G(a) = \{1, 2\}$. Let $\tau = \{0_A, 1_A, F\}$. Then $G \in SO(X, \tau, A) - \omega_s(X, \tau, A)$.

Theorem 2.6. If (X, τ, A) is a soft anti-locally countable STS, then $\omega_s(X, \tau, A) = SO(X, \tau, A)$.

Proof. By Theorem 2.2, we only need to see that $SO(X, \tau, A) \subseteq \omega_s(X, \tau, A)$. Let $K \in SO(X, \tau, A)$, then there is $F \in \tau$ such that $F \subseteq K \subseteq Cl_\tau(F)$. By Proposition 1.4, $Cl_\tau(F) = Cl_{\tau_\omega}(F)$. Therefore, $K \in \omega_s(X, \tau, A)$. This shows that $SO(X, \tau, A) \subseteq \omega_s(X, \tau, A)$. $\square$

Theorem 2.7. For any STS (X, τ, A) , $SO(X, \tau_\omega, A) = \omega_s(X, \tau_\omega, A)$.

Proof. Let (X, τ, A) be STS. By Theorem 2.2, we only need to show that $SO(X, \tau_\omega, A) \subseteq \omega_s(X, \tau_\omega, A)$. Let $K \in$

$SO(X, \tau_\omega, A)$, then there is $F \in \tau_\omega$ such that $F \subseteq K \subseteq Cl_{(\tau_\omega)_\omega}(F)$. As by Proposition 1.6, $(\tau_\omega)_\omega = \tau_\omega$, then $Cl_{(\tau_\omega)_\omega}(F) = Cl_{\tau_\omega}(F)$. Hence, $K \in \omega_s(X, \tau_\omega, A)$. This ends the proof that $SO(X, \tau_\omega, A) \subseteq \omega_s(X, \tau_\omega, A)$. $\square$

Theorem 2.8. If (X, τ, A) is a soft locally countable STS, then $\tau = \omega_s(X, \tau, A)$.

Proof. By Theorem 2.2, we only need to see that $\omega_s(X, \tau, A) \subseteq \tau$. Let $K \in \omega_s(X, \tau, A)$. Then there is $F \in \tau$ such that $F \subseteq K \subseteq Cl_{\tau_\omega}(F)$. By Proposition 1.5, $Cl_{\tau_\omega}(F) = F$ which implies that $K = F$. Thus, $K \in \tau$. This shows that $\tau = \omega_s(X, \tau, A)$. $\square$

Soft ω -open sets and soft ω_s -open sets are independent of each other:

Example 2.9. Let $X = \mathbb{R}$ and $A = \mathbb{N}$. Let $F, G, K \in SS(X, A)$ defined by $F(a) = \{a\} \cup (2, 3)$, $G(a) = [0, \infty)$ and $K(a) = (2, 3)$ for all $a \in A$. Let $\tau = \{0_A, 1_A, F\}$. As (X, τ, A) is soft anti-locally countable, then by Proposition 1.5, $Cl_{\tau_\omega}(F) = Cl_\tau(F) = 1_A$. So $G \in \omega_s(X, \tau, A) - \tau_\omega$ and $K \in \tau_\omega - \omega_s(X, \tau, A)$.

Theorem 2.10. Let (X, τ, A) be a STS and let $K \in SS(X, A)$. Then K is soft ω_s -open in (X, τ, A) if and only if $K \subseteq Cl_{\tau_\omega}(int_\tau(K))$.

Proof. Necessity. Suppose that K is soft ω_s -open set in (X, τ, A) . Then we find $F \in \tau$ such that $F \subseteq K \subseteq Cl_{\tau_\omega}(F)$. As $F \subseteq K$, then $F = int_\tau(F) \subseteq int_\tau(K)$, which implies that $Cl_{\tau_\omega}(F) \subseteq Cl_{\tau_\omega}(int_\tau(K))$. It follows that $K \subseteq Cl_{\tau_\omega}(int_\tau(K))$.

Sufficiency. Suppose that $K \subseteq Cl_{\tau_\omega}(int_\tau(K))$. Put $F = int_\tau(K)$. Then $F \in \tau$ with $F \subseteq K \subseteq Cl_{\tau_\omega}(int_\tau(K))$. Hence, K is soft ω_s -open set in (X, τ, A) . $\square$

Theorem 2.11. Let (X, τ, A) be a STS. If $\{K_\alpha : \alpha \in \Delta\} \subseteq \omega_s(X, \tau, A)$, then $\bigcup_{\alpha \in \Delta} K_\alpha \in \omega_s(X, \tau, A)$.

Proof. For every $\alpha \in \Delta$, we find $F_\alpha \in \tau$ such that $F_\alpha \subseteq K_\alpha \subseteq Cl_{\tau_\omega}(F_\alpha)$. Then $\bigcup_{\alpha \in \Delta} F_\alpha \in \tau$ and $\bigcup_{\alpha \in \Delta} F_\alpha \subseteq \bigcup_{\alpha \in \Delta} K_\alpha \subseteq \bigcup_{\alpha \in \Delta} Cl_{\tau_\omega}(K_\alpha) \subseteq Cl_{\tau_\omega}\left(\bigcup_{\alpha \in \Delta} K_\alpha\right)$. Therefore, $\bigcup_{\alpha \in \Delta} K_\alpha \in \omega_s(X, \tau, A)$. $\square$

Soft intersection of two soft ω_s -open sets is not a soft ω_s -open in general:

Example 2.12. Let $X = \mathbb{R}$ and $A = \mathbb{N}$. Let $F, G \in SS(X, A)$ defined by $F(a) = (0, \infty)$ and $G(a) = (-\infty, 0)$ for all $a \in A$. Let $\tau = \{0_A, 1_A, F, G, F \cup G\}$. As (X, τ, A) is soft anti-locally countable, then by Proposition 1.4, $Cl_{\tau_\omega}(F) = Cl_\tau(F) = 1_A - G$ and $Cl_{\tau_\omega}(G) = Cl_\tau(G) = 1_A - F$. Then $1_A - F, 1_A - G \in \omega_s(X, \tau, A)$ but $((1_A - F) \cap (1_A - G))(a) = \{0\}$ for all $a \in A$ and hence $(1_A - F) \cap (1_A - G) \notin \omega_s(X, \tau, A)$.

Theorem 2.13. Let (X, τ, A) be a STS. If $G \in \tau$ and $K \in \omega_s(X, \tau, A)$, then $G \cap K \in \omega_s(X, \tau, A)$.

Proof. Let $G \in \tau$ and $K \in \omega_s(X, \tau, A)$. As $K \in \omega_s(X, \tau, A)$, then there is $F \in \tau$ such that $F \subseteq K \subseteq Cl_{\tau_\omega}(F)$. Then $G \cap F \in \tau$ and $G \cap F \subseteq G \cap K \subseteq G \cap Cl_{\tau_\omega}(F) \subseteq Cl_{\tau_\omega}(G \cap F)$. This ends the proof that $G \cap K \in \omega_s(X, \tau, A)$. $\square$

Theorem 2.14. Let (X, τ, A) be a STS. If $K \in \omega_s(X, \tau, A)$ and $K \subseteq G \subseteq Cl_{\tau_\omega}(K)$, then $G \in \omega_s(X, \tau, A)$.

Proof. Suppose that $K \in \omega_s(X, \tau, A)$ and $K \subseteq G \subseteq Cl_{\tau_\omega}(K)$. As $K \in \omega_s(X, \tau, A)$, then there is $F \in \tau$ such that $F \subseteq K \subseteq Cl_{\tau_\omega}(F)$. As $K \subseteq Cl_{\tau_\omega}(F)$, then $Cl_{\tau_\omega}(K) \subseteq Cl_{\tau_\omega}(F)$. As $G \subseteq Cl_{\tau_\omega}(K)$, then $G \subseteq Cl_{\tau_\omega}(F)$. Thus, we have $F \in \tau$ and $F \subseteq K \subseteq G \subseteq Cl_{\tau_\omega}(K) \subseteq Cl_{\tau_\omega}(F)$.

Therefore, $G \in \omega_s(X, \tau, A)$. $\square$

Theorem 2.15. For any STS (X, τ, A) ,

$$\tau = \{int_\tau(K) : K \in \omega_s(X, \tau, A)\}.$$

Proof. As $int_\tau(K) \in \tau$ for every $K \in \omega_s(X, \tau, A)$, then $\{int_\tau(K) : K \in \omega_s(X, \tau, A)\} \subseteq \tau$. Conversely, let $F \in \tau$, then $F = int_\tau(F)$. On the other hand, by Theorem 2.2, $F \in \omega_s(X, \tau, A)$. It follows that $\tau \subseteq \{int_\tau(K) : K \in \omega_s(X, \tau, A)\}$. $\square$

Definition 2.16. Let (X, τ, A) be a STS and let $M \in SS(X, A)$. Then M is called soft ω_s -closed set in (X, τ, A) if $1_A - M$ is soft ω_s -open set in (X, τ, A) .

Theorem 2.17. Let (X, τ, A) be a STS. Then

(a) Every soft closed set in (X, τ, A) is soft ω_s -closed set in (X, τ, A) .

(b) Every soft ω_s -closed set in (X, τ, A) is soft semi-closed set in (X, τ, A) .

(c) 0_A and 1_A are soft ω_s -closed sets in (X, τ, A) .

Proof. (a) Let M be a soft closed set in (X, τ, A) , then $1_A - M \in \tau$ and by Theorem 2.2 $1_A - M \in \omega_s(X, \tau, A)$. Thus, M is a soft ω_s -closed set in (X, τ, A) .

(b) Let M be a soft ω_s -closed set in (X, τ, A) , then $1_A - M \in \omega_s(X, \tau, A)$ and by Theorem 2.2 $1_A - M \in SO(X, \tau, A)$. Thus, M is soft semi-closed set in (X, τ, A) .

(c) Follows by (a). $\square$

One can use Example 2.3 to show that none of the implications in Theorem 2.15 (a), (b) is reversible in general.

Theorem 2.18. Let (X, τ, A) be a STS. If M_α is soft ω_s -closed in (X, τ, A) for all $\alpha \in \Delta$, then $\bigcap_{\alpha \in \Delta} M_\alpha$ is soft ω_s -closed in (X, τ, A) .

Proof. Suppose that M_α is soft ω_s -closed in (X, τ, A) for all $\alpha \in \Delta$. Then for every $\alpha \in \Delta$, $1_A - M_\alpha \in \omega_s(X, \tau, A)$.

Thus, by Theorem 2.9, we have $1_A - \left(\bigcap_{\alpha \in \Delta} M_\alpha \right) = \bigcup_{\alpha \in \Delta} (1_A - M_\alpha) \in \omega_s(X, \tau, A)$. It follows that $\bigcap_{\alpha \in \Delta} M_\alpha$ is soft ω_s -closed in (X, τ, A) . $\square$

Soft ω_s -closedness is not closed under finite soft union:

Example 2.19. Consider (X, τ, A) , F , and G as in Example 2.12. As $1_A - F, 1_A - G \in \omega_s(X, \tau, A)$, then F and G are soft ω_s -closed in (X, τ, A) . As $1_A - (F \widetilde{\cup} G) = (1_A - F) \widetilde{\cap} (1_A - G) \notin \omega_s(X, \tau, A)$, then $F \widetilde{\cup} G$ is not soft ω_s -closed in (X, τ, A) .

Theorem 2.20. Let (X, τ, A) be a STS. If M is soft closed in (X, τ, A) and N is soft ω_s -closed in (X, τ, A) , then $M \widetilde{\cup} N$ is soft ω_s -closed in (X, τ, A) .

Proof. Suppose that M is soft closed in (X, τ, A) and N is soft ω_s -closed in (X, τ, A) . Then $1_A - M \in \tau$ and $1_A - N \in \omega_s(X, \tau, A)$. So by Theorem 2.13, $1_A - (M \widetilde{\cup} N) = (1_A - M) \widetilde{\cap} (1_A - N) \in \omega_s(X, \tau, A)$. It follows that $M \widetilde{\cup} N$ is soft ω_s -closed in (X, τ, A) . $\square$

Theorem 2.21. Let (X, τ, A) be a STS and let $M \in SS(X, A)$. Then M is soft ω_s -closed in (X, τ, A) if and only if

$$int_{\tau_\omega}(Cl_\tau(M)) \widetilde{\subseteq} M.$$

Proof. Necessity. Assume that M is soft ω_s -closed in (X, τ, A) . Then $1_A - M$ is soft ω_s -open in (X, τ, A) . Theorem 2.10 implies that $1_A - M \widetilde{\subseteq} Cl_{\tau_\omega}(int_\tau(1_A - M))$, and so

$$\begin{aligned} int_{\tau_\omega}(Cl_\tau(M)) &= Ext_{\tau_\omega}(1_A - Cl_\tau(M)) \\ &= Ext_{\tau_\omega}(Ext_\tau(M)) \\ &= 1_A - Cl_{\tau_\omega}(Ext_\tau(M)) \\ &= 1_A - Cl_{\tau_\omega}(int_\tau(1_A - M)) \\ &\widetilde{\subseteq} M. \end{aligned}$$

Sufficiency. Suppose that $int_{\tau_\omega}(Cl_\tau(M)) \widetilde{\subseteq} M$. Then

$$\begin{aligned} 1_A - M &\subseteq 1_A - int_{\tau_\omega}(Cl_\tau(M)) \\ &= 1_A - Ext_{\tau_\omega}(1_A - Cl_\tau(M)) \\ &= 1_A - Ext_{\tau_\omega}(Ext_\tau(M)) \\ &= Cl_{\tau_\omega}(Ext_\tau(M)) \\ &= Cl_{\tau_\omega}(int_\tau(1_A - M)). \end{aligned}$$

Therefore, by Theorem 2.10, $1_A - M$ is soft ω_s -open in (X, τ, A) and so M is soft ω_s -closed in (X, τ, A) . $\square$

Theorem 2.22. If $f_{pu} : (X, \tau, A) \rightarrow (Y, \sigma, B)$ is a soft open mapping such that $f_{pu} : (X, \tau_\omega, A) \rightarrow (Y, \sigma_\omega, B)$ is

soft continuous, then for every $K \in \omega_s(X, \tau, A)$, we have $f_{pu}(K) \in \omega_s(Y, \sigma, B)$.

Proof. Let $K \in \omega_s(X, \tau, A)$. Then there exists $F \in \tau$ such that $F \widetilde{\subseteq} K \widetilde{\subseteq} Cl_{\tau_\omega}(F)$. Thus we have $f_{pu}(F) \widetilde{\subseteq} f_{pu}(K) \widetilde{\subseteq} f_{pu}(Cl_{\tau_\omega}(F))$. As $f_{pu} : (X, \tau, A) \rightarrow (Y, \sigma, B)$ is soft open mapping, then $f_{pu}(F) \in \sigma$. As $f_{pu} : (X, \tau_\omega, A) \rightarrow (Y, \sigma_\omega, B)$ is soft continuous, then $f_{pu}(Cl_{\tau_\omega}(F)) \widetilde{\subseteq} Cl_{\tau_\omega}(f_{pu}(F))$. Therefore, $f_{pu}(K) \in \omega_s(Y, \sigma, B)$. $\square$

In Theorem 2.22, we cannot drop the condition ‘soft open mapping’:

Example 2.23. Let $X = \mathbb{R}$, $A = \mathbb{N}$, $F, G \in SS(X, A)$ with $F(a) = \mathbb{R}$ and $G(a) = \{0\}$ for all $a \in A$, $\tau = SS(X, A)$, and $\sigma = \{0_B, 1_B, F\}$. Define $p : X \rightarrow X$ and $u : A \rightarrow A$ by $p(x) = 0$ for all $x \in X$ and $u(a) = a$ for all $a \in A$. Then clearly that $f_{pu} : (X, \tau_\omega, A) \rightarrow (X, \tau_\omega, A)$ is soft continuous, but $G \in \omega_s(X, \tau_\omega, A)$ while $f_{pu}(G) = G \notin \omega_s(X, \tau_\omega, A)$.

3. Soft ω_s -Closure and Soft ω_s -Interior

Definition 3.1. Let (X, τ, A) be a STS and $H \in SS(X, A)$. The soft ω_s -closure of H in (X, τ, A) is denoted by $\omega_s\text{-}Cl_\tau(H)$ and defined by $\omega_s\text{-}Cl_\tau(H) = \bigcap \{M : M \text{ is soft } \omega_s\text{-closed in } (X, \tau, A) \text{ and } H \widetilde{\subseteq} M\}$.

Theorem 3.2. Let (X, τ, A) be a STS and $H \in SS(X, A)$. Then

(a) $\omega_s\text{-}Cl_\tau(H)$ is the smallest soft ω_s -closed in (X, τ, A) containing H .

(b) H is soft ω_s -closed in (X, τ, A) if and only if $H = \omega_s\text{-}Cl_\tau(H)$.

Proof. (a) Follows from Definition 3.1 and Theorem 2.18.

(b) Follows immediately by (a). $\square$

Theorem 3.3. Let (X, τ, A) be a STS and $H \in SS(X, A)$. Then $H \widetilde{\subseteq} S\text{-}Cl_\tau(H) \widetilde{\subseteq} \omega_s\text{-}Cl_\tau(H) \widetilde{\subseteq} Cl_\tau(H)$.

Proof. Follows from the definitions and parts (a) and (b) of Theorem 2.17. $\square$

In Theorem 3.3 $S\text{-}Cl_\tau(H) \neq \omega_s\text{-}Cl_\tau(H)$ and $\omega_s\text{-}Cl_\tau(H) \neq Cl_\tau(H)$, in general:

Example 3.4. Consider Example 2.3. We proved that $1_A - G \in SO(X, \tau, A) - \omega_s(X, \tau, A)$ and $1_A - F \in \omega_s(X, \tau, A) - \tau$. So, we have F as an ω_s -closed set in (X, τ, A) that is not closed set in (X, τ, A) , and G as a semi-closed set in (X, τ, A) that is not ω_s -closed set in (X, τ, A) . Thus, by Proposition 3.9 (2) of [26] and Theorem 3.2 (b), $S\text{-}Cl_\tau(G) = G \neq \omega_s\text{-}Cl_\tau(H)$ and $\omega_s\text{-}Cl_\tau(F) = F \neq Cl_\tau(F)$.

Theorem 3.5. Let (X, τ, A) be a STS and $H \in SS(X, A)$. Then

(a) If H is soft closed in (X, τ, A) , then $H = S-Cl_\tau(H) = \omega_s-Cl_\tau(H) = Cl_\tau(H)$.

(b) If H is soft ω_s -closed in (X, τ, A) , then $H = S-Cl_\tau(H) = \omega_s-Cl_\tau(H)$.

Proof. (a) Suppose that soft H is closed in (X, τ, A) . Then $H = Cl_\tau(H)$ and by Theorem 3.3 we have $H = S-Cl_\tau(H) = \omega_s-Cl_\tau(H) = Cl_\tau(H)$.

(b) Suppose that H is soft ω_s -closed in (X, τ, A) . Then by Theorem 3.2 (b) we have $H = \omega_s-Cl_\tau(H)$, and by Theorem 3.3 we conclude that $H = S-Cl_\tau(H) = \omega_s-Cl_\tau(H)$. $\square$

Theorem 3.6. If (X, τ, A) is a soft anti-locally countable STS then for every $H \in SS(X, A)$, $S-Cl_\tau(H) = \omega_s-Cl_\tau(H)$.

Proof. As (X, τ, A) is soft anti-locally countable, then by Theorem 2.6 we have $\omega_s(X, \tau, A) = SO(X, \tau, A)$ and hence $S-Cl_\tau(H) = \omega_s-Cl_\tau(H)$. $\square$

Theorem 3.7. If (X, τ, A) is a soft locally countable STS then for every $H \in SS(X, A)$, $\omega_s-Cl_\tau(H) = Cl_\tau(H)$.

Proof. As (X, τ, A) is soft locally countable, then by Theorem 2.8 we have $\tau = \omega_s(X, \tau, A)$ and hence $\omega_s-Cl_\tau(H) = Cl_\tau(H)$. $\square$

Theorem 3.8. If (X, τ, A) is a STS and $H \in SS(X, A)$, then $S-Cl_{\tau_\omega}(H) = \omega_s-Cl_{\tau_\omega}(H)$.

Proof. By Theorem 2.7 we have $SO(X, \tau_\omega, A) = \omega_s(X, \tau_\omega, A)$ and hence $S-Cl_{\tau_\omega}(H) = \omega_s-Cl_{\tau_\omega}(H)$. $\square$

Theorem 3.9. Let (X, τ, A) be a STS and $H, K \in SS(X, A)$. Then

- (a) If $H \subseteq K$, then $\omega_s-Cl_\tau(H) \subseteq \omega_s-Cl_\tau(K)$.
- (b) $\omega_s-Cl_\tau(H) \cup \omega_s-Cl_\tau(K) \subseteq \omega_s-Cl_\tau(H \cup K)$.
- (c) $\omega_s-Cl_\tau(H \cap K) \subseteq \omega_s-Cl_\tau(H) \cap \omega_s-Cl_\tau(K)$.

Proof. (a) Suppose that $H \subseteq K$. As $K \subseteq \omega_s-Cl_\tau(K)$, then $H \subseteq \omega_s-Cl_\tau(K)$. By Theorem 3.2 (a), $\omega_s-Cl_\tau(K)$ is soft ω_s -closed in (X, τ, A) . Again by Theorem 3.2 (a), $\omega_s-Cl_\tau(H) \subseteq \omega_s-Cl_\tau(K)$.

(b) As $H \subseteq H \cup K$ and $K \subseteq H \cup K$, then by (a), $\omega_s-Cl_\tau(H) \subseteq \omega_s-Cl_\tau(H \cup K)$ and $\omega_s-Cl_\tau(K) \subseteq \omega_s-Cl_\tau(H \cup K)$. Hence $\omega_s-Cl_\tau(H) \cup \omega_s-Cl_\tau(K) \subseteq \omega_s-Cl_\tau(H \cup K)$.

(c) As $H \cap K \subseteq H$ and $H \cap K \subseteq K$, then by (a), $\omega_s-Cl_\tau(H \cap K) \subseteq \omega_s-Cl_\tau(H)$ and $\omega_s-Cl_\tau(H \cap K) \subseteq \omega_s-Cl_\tau(K)$. Hence $\omega_s-Cl_\tau(H \cap K) \subseteq \omega_s-Cl_\tau(H) \cap \omega_s-Cl_\tau(K)$. $\square$

The soft inclusion in Theorem 3.9 (b) cannot be replaced by soft equality in general:

Example 3.10. Consider (X, τ, A) as in Example 2.19. We proved that F and G are soft ω_s -closed in (X, τ, A) while $F \cup G$ is not soft ω_s -closed in (X, τ, A) . Therefore, by Theorem 3.2 (b), $\omega_s-Cl_\tau(F) \cup \omega_s-Cl_\tau(G) = F \cup G$ while $\omega_s-Cl_\tau(F \cup G) \neq F \cup G$.

Theorem 3.11. If (X, τ, A) is a soft locally countable STS, then for any $H, K \in SS(X, A)$ we have $\omega_s-Cl_\tau(H) \cup \omega_s-Cl_\tau(K) = \omega_s-Cl_\tau(H \cup K)$.

Proof. As (X, τ, A) is soft locally countable, then by Theorem 3.7,

$$\begin{aligned} \omega_s-Cl_\tau(H \cup K) &= Cl_\tau(H \cup K) \\ &= Cl_\tau(H) \cup Cl_\tau(K) \\ &= \omega_s-Cl_\tau(H) \cup \omega_s-Cl_\tau(K). \end{aligned} \quad \square$$

Theorem 3.12. Let (X, τ, A) be a STS and let $H, K \in SS(X, A)$. If H is soft closed in (X, τ, A) , then $\omega_s-Cl_\tau(H) \cup \omega_s-Cl_\tau(K) = \omega_s-Cl_\tau(H \cup K)$.

Proof. By Theorem 3.9 (b) we only need to show that $\omega_s-Cl_\tau(H \cup K) \subseteq \omega_s-Cl_\tau(H) \cup \omega_s-Cl_\tau(K)$. As H is soft closed in (X, τ, A) , then by Theorem 3.5 (a), $H = \omega_s-Cl_\tau(H)$. So by Theorem 2.20, $\omega_s-Cl_\tau(H) \cup \omega_s-Cl_\tau(K)$ is ω_s -closed in (X, τ, A) . As $H \cup K \subseteq \omega_s-Cl_\tau(H) \cup \omega_s-Cl_\tau(K)$, then by Theorem 3.2 (a) we have $\omega_s-Cl_\tau(H \cup K) \subseteq \omega_s-Cl_\tau(H) \cup \omega_s-Cl_\tau(K)$. $\square$

The soft inclusion in Theorem 3.9 (c) cannot be replaced by soft equality in general:

Example 3.13. Let $X = \mathbb{Z} \times \mathbb{Z}$, $A = \mathbb{R}$ and $\tau = \{F \in SS(X, A) : F(a) \in \tau_{cof} \text{ for every } a \in A\}$. Let $H, K \in SS(X, A)$ defined by $H(a) = \mathbb{Z} \times \{0\}$ and $K(a) = \{0\} \times \mathbb{Z}$ for all $a \in A$. Then $(H \cap K)(a) = \{(0, 0)\}$ for all $a \in A$. Note that $M \in SS(X, A)$ is soft closed in (X, τ, A) if and only if for all $a \in A$, either $M(a)$ is finite or $M(a) = \mathbb{Z} \times \mathbb{Z}$. Then $Cl_\tau(H) = Cl_\tau(K) = 1_A$ and $Cl_\tau(H \cap K) = H \cap K$. As (X, τ, A) is soft locally countable, then by Theorem 3.7, $\omega_s-Cl_\tau(H) = Cl_\tau(H) = 1_A$, $\omega_s-Cl_\tau(K) = Cl_\tau(K) = 1_A$, and $\omega_s-Cl_\tau(H \cap K) = H \cap K$. Thus, $\omega_s-Cl_\tau(H) \cap \omega_s-Cl_\tau(K) = 1_A$ while $\omega_s-Cl_\tau(H \cap K) \neq 1_A$.

Theorem 3.14. Let (X, τ, A) be a STS and let $H \in SS(X, A)$. Then $S-Cl_\tau(Cl_\tau(H)) = Cl_\tau(S-Cl_\tau(H)) = \omega_s-Cl_\tau(Cl_\tau(H)) = Cl_\tau(\omega_s-Cl_\tau(H)) = Cl_\tau(H)$.

Proof. As $Cl_\tau(H)$ is soft closed in (X, τ, A) , then, by Theorem 3.5 (a) we have $Cl_\tau(H) = S-Cl_\tau(Cl_\tau(H)) = \omega_s-Cl_\tau(Cl_\tau(H))$.

By Theorem 3.3, $H \subseteq S-Cl_\tau(H) \subseteq \omega_s-Cl_\tau(H) \subseteq Cl_\tau(H)$. So, $Cl_\tau(H) \subseteq Cl_\tau(S-Cl_\tau(H)) \subseteq Cl_\tau(\omega_s-Cl_\tau(H)) \subseteq Cl_\tau(Cl_\tau(H)) = Cl_\tau(H)$. Hence, we have $Cl_\tau(H) = Cl_\tau(S-Cl_\tau(H)) = Cl_\tau(\omega_s-Cl_\tau(H))$. $\square$

Theorem 3.15. Let (X, τ, A) be a STS and let $H \in SS(X, A)$. Then $\omega_s-Cl_\tau(S-Cl_\tau(H)) = S-Cl_\tau(\omega_s-Cl_\tau(H)) = \omega_s-Cl_\tau(H)$.

Proof. As $\omega_s\text{-}Cl_\tau(H)$ is soft ω_s -closed in (X, τ, A) , then by Theorem 3.5 (b) we have $\omega_s\text{-}Cl_\tau(H) = S\text{-}Cl_\tau(\omega_s\text{-}Cl_\tau(H))$.

By Theorem 3.3, $H \subseteq S\text{-}Cl_\tau(H) \subseteq \omega_s\text{-}Cl_\tau(H)$. So by Theorem 3.9 (a), $\omega_s\text{-}Cl_\tau(H) \subseteq \omega_s\text{-}Cl_\tau(S\text{-}Cl_\tau(H)) \subseteq \omega_s\text{-}Cl_\tau(\omega_s\text{-}Cl_\tau(H)) = \omega_s\text{-}Cl_\tau(H)$. Therefore,

$$\omega_s\text{-}Cl_\tau(S\text{-}Cl_\tau(H)) = \omega_s\text{-}Cl_\tau(H).$$

□

Theorem 3.16. Let (X, τ, A) be a STS and let $H \in SS(X, A)$. Then $a_x \in \omega_s\text{-}Cl_\tau(H)$ if and only if for all $K \in \omega_s(X, \tau, A)$ with $a_x \in K$ we have $K \cap H \neq \emptyset$.

Proof. Necessity. Suppose that $a_x \in \omega_s\text{-}Cl_\tau(H)$ and suppose to the contrary that there is $K \in \omega_s(X, \tau, A)$ such that $a_x \in K$ and $K \cap H = \emptyset$. So we have $H \subseteq 1_A - K$ with $1_A - K$ is soft ω_s -closed set in (X, τ, A) , and hence $\omega_s\text{-}Cl_\tau(H) \subseteq 1_A - K$. As $a_x \in \omega_s\text{-}Cl_\tau(H)$, then $a_x \in 1_A - K$, a contradiction.

Sufficiency. By contradiction. Then we have $a_x \in (1_A - \omega_s\text{-}Cl_\tau(H)) \in \omega_s(X, \tau, A)$. Then by assumption $(1_A - \omega_s\text{-}Cl_\tau(H)) \cap H = \emptyset$, a contradiction. □

Definition 3.17. Let (X, τ, A) be a STS and $H \in SS(X, A)$. The soft ω_s -interior of H in (X, τ, A) is denoted by $\omega_s\text{-}int_\tau(H)$ and defined by $\omega_s\text{-}int_\tau(H) = \bigcup \{K : K \text{ is soft } \omega_s\text{-open in } (X, \tau, A) \text{ and } K \subseteq H\}$.

Theorem 3.18. Let (X, τ, A) be a STS and $H \in SS(X, A)$. Then

(a) $\omega_s\text{-}int_\tau(H)$ is the largest soft ω_s -open in (X, τ, A) contained in H .

(b) H is soft ω_s -open in (X, τ, A) if and only if $H = \omega_s\text{-}int_\tau(H)$.

Proof. (a) Follows from Definition 3.17 and Theorem 2.11.

(b) Follows immediately by (a). □

Theorem 3.19. Let (X, τ, A) be a STS and $H \in SS(X, A)$. Then $int_\tau(H) \subseteq \omega_s\text{-}int_\tau(H) \subseteq S\text{-}int_\tau(H) \subseteq H$.

Proof. Follows from the definitions and Theorem 2.2. □

In Theorem 3.19 $int_\tau(H) \neq \omega_s\text{-}int_\tau(H)$ and $\omega_s\text{-}int_\tau(H) \neq S\text{-}int_\tau(H)$, in general:

Example 3.20. Consider Example 2.3. We proved that $1_A - G \in SO(X, \tau, A) - \omega_s(X, \tau, A)$ and $1_A - F \in \omega_s(X, \tau, A) - \tau$. Thus, by Proposition 3.9 (2) and Theorem 3.2 (b), $S\text{-}int_\tau(1_A - G) = 1_A - G \neq \omega_s\text{-}int_\tau(1_A - G)$ and $\omega_s\text{-}int_\tau(1_A - F) = 1_A - F \neq int_\tau(1_A - F)$.

Theorem 3.21. Let (X, τ, A) be a STS and $H \in SS(X, A)$. Then

(a) If H is soft open in (X, τ, A) , then $H = S\text{-}int_\tau(H) = \omega_s\text{-}int_\tau(H) = int_\tau(H)$.

(b) If H is soft ω_s -open in (X, τ, A) , then $H = S\text{-}int_\tau(H) =$

$\omega_s\text{-}int_\tau(H)$.

Proof. (a) Suppose that H is soft open in (X, τ, A) . Then $H = int_\tau(H)$ and by Theorem 3.19 we have $H = S\text{-}int_\tau(H) = \omega_s\text{-}int_\tau(H) = int_\tau(H)$.

(b) Suppose that H is soft ω_s -open in (X, τ, A) . Then by Theorem 3.2 (b), $H = \omega_s\text{-}int_\tau(H)$. Thus, by Theorem 3.3 we $\omega_s\text{-}int_\tau(H) \subseteq S\text{-}int_\tau(H) \subseteq \omega_s\text{-}int_\tau(H)$. Therefore, $H = S\text{-}int_\tau(H) = \omega_s\text{-}int_\tau(H)$. □

Theorem 3.22. If (X, τ, A) is a soft anti-locally countable STS then for every $H \in SS(X, A)$, $S\text{-}int_\tau(H) = \omega_s\text{-}int_\tau(H)$.

Proof. As (X, τ, A) is soft anti-locally countable, then by Theorem 2.6 we have $\omega_s(X, \tau, A) = SO(X, \tau, A)$ and hence $S\text{-}int_\tau(H) = \omega_s\text{-}int_\tau(H)$. □

Theorem 3.23. If (X, τ, A) is a soft locally countable STS then for every $H \in SS(X, A)$, $\omega_s\text{-}int_\tau(H) = int_\tau(H)$.

Proof. As (X, τ, A) is soft locally countable, then by Theorem 2.8 we have $\tau = \omega_s(X, \tau, A)$ and hence $\omega_s\text{-}int_\tau(H) = int_\tau(H)$. □

Theorem 3.24. If (X, τ, A) is a STS and $H \in SS(X, A)$, then $S\text{-}int_{\tau_\omega}(H) = \omega_s\text{-}int_{\tau_\omega}(H)$.

Proof. By Theorem 2.7 we have $SO(X, \tau_\omega, A) = \omega_s(X, \tau_\omega, A)$ and hence $S\text{-}int_{\tau_\omega}(H) = \omega_s\text{-}int_{\tau_\omega}(H)$. □

Theorem 3.25. Let (X, τ, A) be a STS and $H, K \in SS(X, A)$. Then

(a) If $H \subseteq K$, then $\omega_s\text{-}int_\tau(H) \subseteq \omega_s\text{-}int_\tau(K)$.

(b) $\omega_s\text{-}int_\tau(H) \cup \omega_s\text{-}int_\tau(K) \subseteq \omega_s\text{-}int_\tau(H \cup K)$.

(c) $\omega_s\text{-}int_\tau(H \cap K) \subseteq \omega_s\text{-}int_\tau(H) \cap \omega_s\text{-}int_\tau(K)$.

Proof. (a) Suppose that $H \subseteq K$. Then we have $\omega_s\text{-}int_\tau(H) \subseteq H \subseteq K$. By Theorem 3.18 (a), $\omega_s\text{-}int_\tau(H)$ is soft ω_s -open in (X, τ, A) . Again by Theorem 3.18 (a), $\omega_s\text{-}int_\tau(H) \subseteq \omega_s\text{-}int_\tau(K)$.

(b) As $H \subseteq H \cup K$ and $K \subseteq H \cup K$, then by (a), $\omega_s\text{-}int_\tau(H) \subseteq \omega_s\text{-}int_\tau(H \cup K)$ and $\omega_s\text{-}int_\tau(K) \subseteq \omega_s\text{-}int_\tau(H \cup K)$. Hence $\omega_s\text{-}int_\tau(H) \cup \omega_s\text{-}int_\tau(K) \subseteq \omega_s\text{-}int_\tau(H \cup K)$.

(c) As $H \cap K \subseteq H$ and $H \cap K \subseteq K$, then by (a), $\omega_s\text{-}int_\tau(H \cap K) \subseteq \omega_s\text{-}int_\tau(H)$ and $\omega_s\text{-}int_\tau(H \cap K) \subseteq \omega_s\text{-}int_\tau(K)$. Hence $\omega_s\text{-}int_\tau(H \cap K) \subseteq \omega_s\text{-}int_\tau(H) \cap \omega_s\text{-}int_\tau(K)$. □

The soft inclusion in Theorem 3.26 (c) cannot be replaced by soft equality in general:

Example 3.27. Consider (X, τ, A) as in Example 2.12. We proved that $1_A - F$ and $1_A - G$ are soft ω_s -open in (X, τ, A) while $(1_A - F) \cap (1_A - G)$ is not soft ω_s -open in (X, τ, A) . Therefore, by Theorem 3.18 (b),

$$\omega_s\text{-}int_\tau(1_A - F) \cap \omega_s\text{-}int_\tau(1_A - G) = (1_A - F) \cap (1_A - G),$$

while

$$\omega_s\text{-int}_\tau((1_A - F) \widetilde{\cap} (1_A - G)) \neq (1_A - F) \widetilde{\cap} (1_A - G). \quad \square$$

Theorem 3.28. If (X, τ, A) is a soft locally countable STS, then for any $H, K \in SS(X, A)$ we have $\omega_s\text{-int}_\tau(H) \widetilde{\cap} \omega_s\text{-int}_\tau(K) = \omega_s\text{-int}_\tau(H \widetilde{\cap} K)$.

Proof. As (X, τ, A) is soft locally countable, then by Theorem 3.23,

$$\begin{aligned} \omega_s\text{-int}_\tau(H \widetilde{\cap} K) &= \text{int}_\tau(H \widetilde{\cap} K) \\ &= \text{Cl}_\tau(H) \widetilde{\cap} \text{Cl}_\tau(K) \\ &= \omega_s\text{-Cl}_\tau(H) \widetilde{\cap} \omega_s\text{-Cl}_\tau(K). \end{aligned} \quad \square$$

Theorem 3.29. Let (X, τ, A) be a STS and let $H, K \in SS(X, A)$. If H is soft open in (X, τ, A) , then $\omega_s\text{-int}_\tau(H) \widetilde{\cap} \omega_s\text{-int}_\tau(K) = \omega_s\text{-int}_\tau(H \widetilde{\cap} K)$.

Proof. By Theorem 3.25 (c) we only need to show that $\omega_s\text{-int}_\tau(H) \widetilde{\cap} \omega_s\text{-int}_\tau(K) \subseteq \omega_s\text{-int}_\tau(H \widetilde{\cap} K)$. As H is soft open in (X, τ, A) , then by Theorem 3.18 (b), $H = \omega_s\text{-int}_\tau(H)$. So by Theorem 2.13, $\omega_s\text{-int}_\tau(H) \widetilde{\cap} \omega_s\text{-int}_\tau(K)$ is soft ω_s -open in (X, τ, A) . As $\omega_s\text{-int}_\tau(H) \widetilde{\cap} \omega_s\text{-int}_\tau(K) \subseteq H \widetilde{\cap} K$, then by Theorem 3.18 (a) we have $\omega_s\text{-int}_\tau(H) \widetilde{\cap} \omega_s\text{-int}_\tau(K) \subseteq \omega_s\text{-int}_\tau(H \widetilde{\cap} K)$. $\square$

The soft inclusion in Theorem 3.25 (b) cannot be replaced by soft equality in general:

Example 3.30. Consider (X, τ, A) in Example 3.13 and let $B = \{-n : n \in \mathbb{N}\}$. Let $H, K \in SS(X, A)$ defined by $H(a) = \mathbb{Z} \times \mathbb{N}$ and $K(a) = (\{0\} \cup B) \times \mathbb{Z}$ for all $a \in A$. As (X, τ, A) is soft locally countable, then by Theorem 3.23, $\omega_s\text{-int}_\tau(H) = \text{int}_\tau(H) = 0_A$, $\omega_s\text{-int}_\tau(K) = \text{int}_\tau(K) = 0_A$, and $\omega_s\text{-Cl}_\tau(H \widetilde{\cup} K) = \text{int}_\tau(H \widetilde{\cup} K) = \text{int}_\tau(1_A) = 1_A$. Thus, $\omega_s\text{-int}_\tau(H) \widetilde{\cup} \omega_s\text{-int}_\tau(K) = 0_A$ while $\omega_s\text{-Cl}_\tau(H \widetilde{\cup} K) \neq 0_A$.

Theorem 3.31. Let (X, τ, A) be a STS and let $H \in SS(X, A)$. Then $\text{int}_\tau(H) = \omega_s\text{-int}_\tau(\text{int}_\tau(H)) = S\text{-int}_\tau(\text{int}_\tau(H)) = \text{int}_\tau(\omega_s\text{-int}_\tau(H)) = \text{int}_\tau(S\text{-int}_\tau(H))$.

Proof. As $\text{int}_\tau(H)$ is soft open in (X, τ, A) , then by Theorem 3.21 (a) we have $\text{int}_\tau(H) = S\text{-int}_\tau(\text{int}_\tau(H)) = \omega_s\text{-int}_\tau(\text{int}_\tau(H))$.

By Theorem 3.19 we have $\text{int}_\tau(H) \subseteq \omega_s\text{-int}_\tau(H) \subseteq S\text{-int}_\tau(H) \subseteq H$. So, $\text{int}_\tau(H) = \text{int}_\tau(\text{int}_\tau(H)) \subseteq \text{int}_\tau(\omega_s\text{-int}_\tau(H)) \subseteq \text{int}_\tau(S\text{-int}_\tau(H)) \subseteq \text{int}_\tau(H)$.

Hence,

$$\text{int}_\tau(H) = \text{int}_\tau(\omega_s\text{-int}_\tau(H)) = \text{int}_\tau(S\text{-int}_\tau(H)). \quad \square$$

Theorem 3.32. Let (X, τ, A) be a STS and let $H \in SS(X, A)$. Then $\omega_s\text{-int}_\tau(S\text{-int}_\tau(H)) = S\text{-int}_\tau(\omega_s\text{-int}_\tau(H)) = \omega_s\text{-int}_\tau(H)$.

Proof. As $\omega_s\text{-int}_\tau(H)$ is soft ω_s -open in (X, τ, A) , then by Theorem 3.21 (b) we have $\omega_s\text{-int}_\tau(H) = S\text{-int}_\tau(\omega_s\text{-int}_\tau(H))$. By Theorem 3.19 we have

$$\omega_s\text{-int}_\tau(H) \subseteq S\text{-int}_\tau(H) \subseteq H.$$

So by Theorem 3.25 (a) we have $\omega_s\text{-int}_\tau(H) = \omega_s\text{-int}_\tau(\omega_s\text{-int}_\tau(H)) \subseteq \omega_s\text{-int}_\tau(S\text{-int}_\tau(H)) \subseteq \omega_s\text{-int}_\tau(H)$. Therefore, $\omega_s\text{-int}_\tau(H) = \omega_s\text{-int}_\tau(S\text{-int}_\tau(H))$. $\square$

Theorem 3.33. Let (X, τ, A) be a STS and let $H \in SS(X, A)$. Then

- (a) $\omega_s\text{-int}_\tau(1_A - H) \widetilde{\cap} \omega_s\text{-Cl}_\tau(H) = 0_A$.
- (b) $1_A = \omega_s\text{-int}_\tau(1_A - H) \widetilde{\cup} \omega_s\text{-Cl}_\tau(H)$.
- (c) $1_A - \omega_s\text{-Cl}_\tau(H) = \omega_s\text{-int}_\tau(1_A - H)$ and $1_A - \omega_s\text{-int}_\tau(1_A - H) = \omega_s\text{-Cl}_\tau(H)$.

Proof. (a) Suppose to the contrary that there exists $a_x \in (\omega_s\text{-int}_\tau(1_A - H) \widetilde{\cap} \omega_s\text{-Cl}_\tau(H))$. Then we have $a_x \in \omega_s\text{-int}_\tau(1_A - H) \in \omega_s(X, \tau, A)$ and $a_x \in \omega_s\text{-Cl}_\tau(H)$. Thus by Theorem 3.16 $\omega_s\text{-int}_\tau(1_A - H) \widetilde{\cap} H \neq 0_A$. But $\omega_s\text{-int}_\tau(1_A - H) \widetilde{\cap} H \subseteq (1_A - H) \widetilde{\cap} H = 0_A$, a contradiction.

(b) It is sufficient to show that

$$1_A - \omega_s\text{-Cl}_\tau(H) \subseteq \omega_s\text{-int}_\tau(1_A - H).$$

Let $a_x \in (1_A - \omega_s\text{-Cl}_\tau(H))$, then by Theorem 3.16 there is $K \in \omega_s(X, \tau, A)$ such that $a_x \in K$ and $K \widetilde{\cap} H = 0_A$. So, we have $a_x \in K \subseteq 1_A - H$ and hence $a_x \in \omega_s\text{-int}_\tau(1_A - H)$.

(c) Follows from (a) and (b). $\square$

4. Relative Topological Spaces and Generated Soft Topological Spaces

At this stage, we believe that the following four questions are natural:

Question 4.1. Let (X, τ, A) and let $K \in \omega_s(X, \tau, A)$. Is it true that $K(a) \in \omega_s(X, \tau_a)$ for all $a \in A$?

Question 4.2. Let (X, τ, A) and let $K \in SS(X, A)$ such that $K(a) \in \omega_s(X, \tau_a)$ for all $a \in A$. Is it true that $K \in \omega_s(X, \tau, A)$?

Question 4.3. Let (X, τ, A) and let $K \in SO(X, \tau, A)$. Is it true that $K(a) \in SO(X, \tau_a)$ for all $a \in A$?

Question 4.4. Let (X, τ, A) and let $K \in SS(X, A)$ such that $K(a) \in SO(X, \tau_a)$ for all $a \in A$. Is it true that $K \in SO(X, \tau, A)$?

We will leave Question 4.3 as an open question. However, in the following example we will give negative answers for Questions 4.1, 4.2, and 4.4:

Example 4.5. Let $X = \mathbb{R}$ and $A = \{a, b\}$. Let $F, G, H, K \in SS(X, A)$ defined by

$$\begin{aligned} F(a) &= \mathbb{Q}^c, \quad F(b) = \{1\}, \\ G(a) &= \{2\}, \quad G(b) = \mathbb{Q}^c, \\ H(a) &= \mathbb{Q}^c, \quad H(b) = \mathbb{N}, \\ K(a) &= \{2\}, \quad K(b) = \{1\}. \end{aligned}$$

Let $\tau = \{0_A, 1_A, F, G, F \cup G\}$. As clearly that (X, τ, A) is soft anti-locally countable, then by Theorem 2.6 $\omega_s(X, \tau, A) = SO(X, \tau, A)$. It is not difficult to check that $H \in SO(X, \tau, A)$ while $H(b) \notin \omega_s(X, \tau_b)$, which gives a negative answer for Questions 4.1. Also it is not difficult to check that $K(a) \in \omega_s(X, \tau_a) \subseteq SO(X, \tau_a)$ and $K(b) \in \omega_s(X, \tau_b) \subseteq SO(X, \tau_b)$ while $K \notin \omega_s(X, \tau, A) = SO(X, \tau, A)$, which gives negative answers for Questions 4.2 and 4.4.

If we add the condition ‘ (X, τ, A) is soft locally countable’, then Question 4.1 will have a positive answer:

Theorem 4.6. Let (X, τ, A) be a soft locally countable and let $K \in \omega_s(X, \tau, A)$. Then $K(a) \in \omega_s(X, \tau_a)$ for every $a \in A$.

Proof. Let $a \in A$. As (X, τ, A) is soft locally countable and $K \in \omega_s(X, \tau, A)$, then by Theorem 2.8, $K \in \tau$ and so $K(a) \in \tau_a \subseteq \omega_s(X, \tau_a)$. $\square$

Lemma 4.7. Let $\{(X, \mathfrak{S}_a) : a \in A\}$ be an indexed family of topological spaces and let $\tau = \bigoplus_{a \in A} \mathfrak{S}_a$. Let $H \in SS(X, A)$, then for every $a \in A$, $Cl_{(\tau_a)_\omega}(H(a)) = (Cl_{\tau_\omega}(H))(a)$.

Proof. Let $a \in A$. By Proposition 7 of [4], $Cl_{(\tau_a)_\omega}(H(a)) \subseteq (Cl_{\tau_\omega}(H))(a)$. To see that $(Cl_{\tau_\omega}(H))(a) \subseteq Cl_{(\tau_a)_\omega}(H(a))$, let $x \in (Cl_{\tau_\omega}(H))(a)$ and let $U \in (\tau_a)_\omega$ such that $x \in U$. By Theorem 7 of [2], $(\tau_a)_\omega = (\tau_\omega)_a$. Then $a_U \in \tau_\omega$. As $a_x \in a_U$ and $a_x \in Cl_{\tau_\omega}(H)$, then by Lemma 5 of [2], $H \cap a_U \neq \emptyset$. Thus, we must have $(H \cap a_U)(a) = H(a) \cap U \neq \emptyset$. It follows that $x \in Cl_{(\tau_a)_\omega}(H(a))$. $\square$

If we add the condition ‘ τ is a generated soft topology’, then Questions 4.1 and 4.2 will have positive answers:

Theorem 4.8. Let $\{(X, \mathfrak{S}_a) : a \in A\}$ be an indexed family of topological spaces and let $\tau = \bigoplus_{a \in A} \mathfrak{S}_a$. Let $H \in SS(X, A)$. Then $H \in \omega_s(X, \tau, A)$ if and only if $H(a) \in \omega_s(X, \tau_a)$ for every $a \in A$.

Proof. Necessity. Suppose that $H \in \omega_s(X, \tau, A)$ and let $a \in A$. Choose $F \in \tau$ such that $F \subseteq H \subseteq Cl_{\tau_\omega}(F)$. So,

$F(a) \subseteq H(a) \subseteq (Cl_{\tau_\omega}(F))(a)$. As $F(a) \in \tau_a$ and by Lemma 4.7 we have $(Cl_{\tau_\omega}(F))(a) = Cl_{(\tau_a)_\omega}(F(a))$, then $H(a) \in \omega_s(X, \tau_a)$.

Sufficiency. Suppose that $H(a) \in \omega_s(X, \tau_a)$ for every $a \in A$. Then for every $a \in A$, there exists $U_a \in \tau_a = \mathfrak{S}_a$ such that $U_a \subseteq H(a) \subseteq Cl_{(\tau_a)_\omega}(U_a)$. Let $K \in SS(X, A)$ with $K(a) = U_a \in \mathfrak{S}_a$ for every $a \in A$. Then $K \in \bigoplus_{a \in A} \mathfrak{S}_a = \tau$ and by Lemma 4.7, $(Cl_{\tau_\omega}(K))(a) = Cl_{(\tau_a)_\omega}(K(a)) = Cl_{(\tau_a)_\omega}(U_a)$ for all $a \in A$. As for every $a \in A$, $K(a) = U_a \subseteq H(a) \subseteq Cl_{(\tau_a)_\omega}(U_a) = (Cl_{\tau_\omega}(K))(a)$, then $K \subseteq H \subseteq Cl_{\tau_\omega}(K)$. It follows that $H \in \omega_s(X, \tau, A)$. $\square$

Lemma 4.9. Let $\{(X, \mathfrak{S}_a) : a \in A\}$ be an indexed family of topological spaces and let $\tau = \bigoplus_{a \in A} \mathfrak{S}_a$. Let $H \in SS(X, A)$, then for every $a \in A$, $Cl_{\tau_a}(H(a)) = (Cl_\tau(H))(a)$.

Proof. Let $a \in A$. By Proposition 7 of [4], $Cl_{\tau_a}(H(a)) \subseteq (Cl_\tau(H))(a)$. To see that $(Cl_\tau(H))(a) \subseteq Cl_{\tau_a}(H(a))$, let $x \in (Cl_\tau(H))(a)$ and let $U \in \tau_a = \mathfrak{S}_a$ such that $x \in U$. Then we have $a_x \in a_U \in \tau$ and $a_x \in Cl_\tau(H)$, and by Lemma 5 of [2], $H \cap a_U \neq \emptyset$. Thus, we must have $(H \cap a_U)(a) = H(a) \cap U \neq \emptyset$. It follows that $x \in Cl_{\tau_a}(H(a))$. $\square$

If we add the condition ‘ τ is a generated soft topology’, then Questions 4.3 and 4.4 will have positive answers:

Theorem 4.10. Let $\{(X, \mathfrak{S}_a) : a \in A\}$ be an indexed family of topological spaces and let $\tau = \bigoplus_{a \in A} \mathfrak{S}_a$. Let $H \in SS(X, A)$. Then $H \in SO(X, \tau, A)$ if and only if $H(a) \in SO(X, \tau_a)$ for every $a \in A$.

Proof. Necessity. Suppose that $H \in SO(X, \tau, A)$. Then there exists $F \in \tau$ such that $F \subseteq H \subseteq Cl_\tau(F)$. So, $F(a) \subseteq H(a) \subseteq (Cl_\tau(F))(a)$. As $F(a) \in \tau_a$ and by Lemma 4.9 we have $(Cl_\tau(F))(a) = Cl_{\tau_a}(F(a))$, then $H(a) \in SO(X, \tau_a)$.

Sufficiency. Suppose that $H(a) \in SO(X, \tau_a)$ for every $a \in A$. Then for every $a \in A$, there exists $U_a \in \tau_a = \mathfrak{S}_a$ such that $U_a \subseteq H(a) \subseteq Cl_{\tau_a}(U_a)$. Let $K \in SS(X, A)$ with $K(a) = U_a \in \mathfrak{S}_a$ for every $a \in A$. Then $K \in \bigoplus_{a \in A} \mathfrak{S}_a = \tau$ and by Lemma 4.9, $(Cl_\tau(K))(a) = Cl_{\tau_a}(K(a)) = Cl_{\tau_a}(U_a)$ for all $a \in A$. As for every $a \in A$, $K(a) = U_a \subseteq H(a) \subseteq Cl_{\tau_a}(U_a) = (Cl_\tau(K))(a)$, then $K \subseteq H \subseteq Cl_\tau(K)$. It follows that $H \in SO(X, \tau, A)$. $\square$

5. Conclusion

We introduced soft ω_s -open sets as a new class of soft sets. We then proved that the class of soft ω_s -open sets lies strictly between the classes of soft open sets and soft semi-open sets. Furthermore, we studied two new soft operators, soft ω_s -closure

and soft ω_s -interior, as well as relationships regarding relative topological spaces and generated soft topological. Finally, we end by raising an open question related to soft ω_s -open. We believe the proposed concept can open up doors to research areas such as soft ω_s -mappings and soft ω_s -compactness.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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