An Improved Bushell-Okarasinski Type Inequality for Sugeno Integrals

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Abstract

Recently, Roman-Flores et al. (2008) proposed a Bushell-Okarasinski type inequality for fuzzy integrals. In this paper, we improve the result of Roman-Flores et al. by findings the optimal constant H_s for which the following Bushell-Okarasinski type inequality for fuzzy integrals $H_s \left((S) \int_0^1 (1-t)^{s-1} g(t)^s d\mu \right) \geq \left((S) \int_0^1 g(t) d\mu \right)^s$ holds, where $s \geq 1$, $g : [0, 1] \rightarrow [0, \infty)$ is a non-increasing function. The case of nondecreasing function is also treated. The result of Roman-Flores et al. is a special case of our result.

Keywords: Fuzzy measure, Sugeno integral, Bushell-Okarasinski type inequality

1. Introduction and Preliminaries

A number of studies have examined the Sugeno integral since its introduction in 1974 [1]. Ralescu and Adams [2] generalized a range of fuzzy measures and gave several equivalent definitions of fuzzy integrals. Wang and Klir [3] provided an overview of fuzzy measure theory.

Caballero and Sadarangani [4–6] proved a Hermite-Hadamard type inequality, a Cauchy type inequality, and Fritz Carlson's inequality for fuzzy integrals. Roman-Flores et al. [7–10] presented several new types of inequalities for Sugeno integrals, including a Prekopa-Leindler type inequality, a Jensen type inequality, a Young type inequality, and some convolution type inequalities. Flores-Franulic et al. [11, 12] presented Chebyshev's inequality and Stolarsky's inequality for fuzzy integrals. Ouyang and Fang [13] generalized their main results to prove some optimal upper bounds for the Sugeno integral of the monotone function in [9]. Ouyang et al. [14] generalized a Chebyshev type inequality for the fuzzy integral of monotone functions based on an arbitrary fuzzy measure. Hong [15] extended previous research by presenting a Hardy-type inequality for Sugeno integrals, Hong [16] proposed a Liapunov type inequality for Sugeno integrals, and Hong [17] proposed a Berward and Favard type inequalities for fuzzy integrals. Hong et al. [19] considered Steffensen's integral inequality for Sugeno integral. Recently, Roman-Flores et al. [10] showed a Bushell-Okarasinski type inequality for fuzzy integrals, but this inequality is not optimal. In this paper, we improve the results of Roman-Flores et al. [10] for $s \geq 1$. Specifically, we find an optimal constant for which a Bushell-Okarasinski type inequality for Sugeno integrals holds for non-increasing functions.

Definition 1. Let Σ be a σ -algebra of subsets of $\mathbb{R}$ and let $\mu : \Sigma \rightarrow [0, \infty]$ be a non-negative, extended real-valued set function. We say that μ is a fuzzy measure if and only if

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- (a) $\mu(\emptyset) = 0$.
- (b) $E, F \in \Sigma$ and $E \subseteq F$ imply $\mu(E) \leq \mu(F)$ (monotonicity).
- (c) $\{E_p\} \subseteq \Sigma$ and $E_1 \subseteq E_2 \subseteq \dots$ imply $\lim_{p \rightarrow \infty} \mu(E_p) = \mu\left(\bigcup_{p=1}^{\infty} E_p\right)$ (continuity from below).
- (d) $\{E_p\} \subseteq \Sigma$, $E_1 \supseteq E_2 \supseteq \dots$, and $\mu(E_1) < \infty$ imply $\lim_{p \rightarrow \infty} \mu(E_p) = \mu\left(\bigcap_{p=1}^{\infty} E_p\right)$ (continuity from above).

If f is a non-negative real-valued function defined on $\mathbb{R}$, then we denote by $F_\alpha = \{x \in X | f(x) \geq \alpha\} = \{f \geq \alpha\}$ the α -level of f , for $\alpha > 0$, and $F_0 = \{x \in X | f(x) > 0\} = \text{supp}(f)$ is the support of f .

We note that

$$\alpha \leq \beta \Rightarrow \{f \geq \beta\} \subseteq \{f \geq \alpha\}.$$

If μ is a fuzzy measure on $A \subset \mathbb{R}$, then we define the following:

$$\mathfrak{F}^\mu(A) = \{f : A \rightarrow [0, \infty) | f \text{ is } \mu\text{-measurable}\}.$$

Definition 2. Let μ be a fuzzy measure on $(\mathbb{R}, \Sigma)$. If $f \in \mathfrak{F}^\mu(\mathbb{R})$ and $A \in \Sigma$, then the Sugeno integral (or the fuzzy integral) of f on A , with respect to the fuzzy measure μ , is defined as

$$(S) \int_A f d\mu = \sup_{\alpha \in [0, \infty)} [\alpha \wedge \mu(A \cap F_\alpha)].$$

In particular, if $A = X$ then

$$(S) \int_{\mathbb{R}} f d\mu = (S) \int f d\mu = \sup_{\alpha \in [0, \infty)} [\alpha \wedge \mu(F_\alpha)].$$

The following properties of the Sugeno integral are well known and can be found in [3]:

Proposition 1 [3]. If μ is a fuzzy measure on $\mathbb{R}$ and $f, g \in \mathfrak{F}^\mu(\mathbb{R})$, then

- (i) $(S) \int_A f d\mu \leq \mu(A)$;
- (ii) $(S) \int_A K d\mu = K \wedge \mu(A)$ for any constant $K \in [0, \infty)$;
- (iii) $(S) \int_A f d\mu \leq (S) \int_A g d\mu$, if $f \leq g$ on A ;
- (iv) $\mu(A \cap \{f \geq \alpha\}) \geq \alpha \Rightarrow (S) \int_A f d\mu \geq \alpha$;
- (v) $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Rightarrow (S) \int_A f d\mu \leq \alpha$;

- (vi) $(S) \int_A f d\mu < \alpha \Leftrightarrow$ there exists $\gamma < \alpha$ such that

$$\mu(A \cap \{f \geq \gamma\}) < \alpha;$$

- (vii) $(S) \int_A f d\mu > \alpha \Leftrightarrow$ there exists $\gamma > \alpha$ such that

$$\mu(A \cap \{f \geq \gamma\}) > \alpha.$$

Note 1. Let $F(\alpha) = \mu(A \cap \{f \geq \alpha\})$. Then by Proposition 1, (iv), (v),

$$F(\alpha) = \alpha \Rightarrow (S) \int_0^1 f(x) d\mu = \alpha.$$

Theorem 1 [13]. Let $f : [0, \infty) \rightarrow [0, \infty)$ be continuous and non-increasing or non-decreasing functions and μ be the Lebesgue measure on $\mathbb{R}$. Let $(S) \int_0^a f(x) d\mu = p$. If $0 < p < a$, then $f(p) = p$ and $f(a - p) = p$, respectively.

2. Bushell-Okrasinski Type Inequality

The classical Bushell-Okrasinski inequality provides the following inequality [19]:

$$\int_0^x (x - t)^{s-1} g(t)^s d\mu \leq \left(\int_0^x g(t) d\mu \right)^s, \quad 0 \leq x \leq b,$$

where $1 \leq s$, $g : [0, 1] \rightarrow [0, \infty)$ is a continuous and increasing function. After changing the variable $t = sx$, Malamud [20] analyzed the Bushell-Okrasinski inequality in the following new form:

$$s \int_0^1 (1 - t)^{s-1} g(t)^s d\mu \leq \left(\int_0^1 g(t) d\mu \right)^s. \quad (1)$$

Recently, Roman-Flores et al. [10] showed a Bushell-Okrasinski inequality derived from (1) for Sugeno integrals as follows:

Theorem 2 (Roman-Flores et al.) [10]. Let $g : [0, 1] \rightarrow [0, \infty)$ be a continuous decreasing function and let μ be the Lebesgue measure on $\mathbb{R}$. Then

$$s(S) \int_0^1 (1 - t)^{s-1} g(t)^s d\mu \geq \left((S) \int_0^1 g(t) d\mu \right)^s$$

holds for all $2 \leq s$.

We now improve the above result.

Theorem 3 (Fuzzy Bushell-Okrasinski inequality). Let $g : [0, 1] \rightarrow [0, \infty)$ be a continuous non-increasing function and let μ be the Lebesgue measure on $\mathbb{R}$. Then

$$\left((S) \int_0^1 (1 - t)^{s-1} d\mu \right)^{-1} \left((S) \int_0^1 (1 - t)^{s-1} g(t)^s d\mu \right)$$

$$\geq \left((S) \int_0^1 g(t) d\mu \right)^s$$

holds for all $1 \leq s$.

Proof. Let

$$H_s = \sup \left\{ \frac{\left((S) \int_0^1 g(t) d\mu \right)^s}{(S) \int_0^1 (1-t)^{s-1} g(t)^s d\mu} \mid g \in \mathcal{G} \right\},$$

where $\mathcal{G}$ is the set of functions which are non-increasing on $[0, 1]$ and let

$$H_s(\alpha) = \sup \left\{ \frac{\alpha^s}{(S) \int_0^1 (1-t)^{s-1} g(t)^s d\mu} \mid g \in \mathcal{G}_\alpha \right\},$$

where $\mathcal{G}_\alpha = \left\{ g \in \mathcal{G} \mid (S) \int_0^1 g(t) d\mu = \alpha \right\}$ for $\alpha \in [0, 1]$. Then

$$H_s = \sup_{0 < \alpha \leq 1} H_s(\alpha).$$

We consider $H_s(\alpha)$. Let

$$g_0(x) = \begin{cases} \alpha, & \text{if } x \in [0, \alpha), \\ 0, & \text{if } x \in [\alpha, 1]. \end{cases}$$

Because $\mu\{g_0 \geq \alpha\} = \alpha$, by Note 1

$$(S) \int_0^1 g_0(x) d\mu = \alpha.$$

Then it is easy to check that $g_0 = \inf \mathcal{G}_\alpha$. Thus, we have

$$H_s(\alpha) = \frac{\alpha^s}{(S) \int_0^1 (1-t)^{s-1} g_0(t)^s d\mu}.$$

Now, let

$$(S) \int_0^1 (1-t)^{s-1} g_0(t)^s d\mu = t_0.$$

Because g_0 is continuous and decreasing on $[0, \alpha]$ and the left limit of g_0 at α is less than α , by Theorem 1,

$$t_0 = (1 - t_0)^{s-1} \alpha^s.$$

To find H_s we now consider the following optimization problem:

$$H_s = \sup_{0 < \alpha \leq 1} H_s(\alpha) = \text{Maximize } \frac{\alpha^s}{t}, \quad (2)$$

where

$$t = (1 - t)^{s-1} \alpha^s, \quad 0 < \alpha \leq 1.$$

We first show that $H_s(\alpha) = \frac{\alpha^s}{t}$ is an increasing function of α . We have

$$H'_s(\alpha) = \frac{\alpha^{s-1}}{t^2} \left(st - \alpha \frac{dt}{d\alpha} \right).$$

From the equation $t = (1 - t)^{s-1} \alpha^s$, we now have

$$\begin{aligned} s\alpha^{s-1} &= \frac{(1-t)^{s-1} + t(s-1)(1-t)^{s-2}}{(1-t)^{2s-2}} \frac{dt}{d\alpha} \\ &= \frac{1 + t(s-1)(1-t)^{-1}}{(1-t)^{s-1}} \frac{dt}{d\alpha}, \end{aligned}$$

which implies that

$$\alpha \frac{dt}{d\alpha} = \frac{1}{1 + t(s-1)(1-t)^{-1}} st \leq st.$$

Then $H'_s(\alpha) \geq 0$, that is, H_s is nondecreasing and thus

$$H_s = H_s(1) = \frac{1}{(S) \int_0^1 (1-t)^{s-1} d\mu},$$

which completes the proof. $\square$

Note 2. As shown in the proof of Theorem 3, the continuity assumption of g is not needed.

Note 3. If $g(t) = 1$ in the inequality of Theorem 3, then the equality holds. Therefore, we see that the inequality in Theorem 3 is optimal.

Lemma 1. Let $(S) \int_0^1 (1-t)^{s-1} d\mu = x^*$, $s \geq 2$. Then

$$\frac{1}{s} \leq \frac{1}{s} \left(\frac{2(s-1)^{s-1}}{(s-1)^{s-1} + s^{s-2}} \right) < x^*.$$

Proof. Suppose that $f(t) = (1-t)^{s-1} - t$. Then by Theorem 1, $f(x^*) = 0$. We note that

$$f'(t) = -(s-1)(1-t)^{s-2} - 1 < 0, \quad f'(0) = -s,$$

and

$$f''(t) = (s-1)(s-2)(1-t)^{s-3} \geq 0.$$

Because f is decreasing and convex, we have

$$\frac{1}{s} = -\frac{f(0)}{f'(0)} < x^*,$$

and similarly, we have

$$\frac{1}{s} \leq \frac{1}{s} - \frac{f(\frac{1}{s})}{f'(\frac{1}{s})} < x^*$$

and

$$\begin{aligned} \frac{1}{s} - \frac{f(\frac{1}{s})}{f'(\frac{1}{s})} &= \frac{1}{s} - \frac{(1 - \frac{1}{s})^{s-1} - \frac{1}{s}}{1 - (s-1)(1 - \frac{1}{s})^{s-2}} \\ &= \frac{1}{s} \left(\frac{2(s-1)^{s-1}}{(s-1)^{s-1} + s^{s-2}} \right), \end{aligned}$$

which completes the proof. $\square$

The following result of Roman-Flores et al. [10] is a special case of our results.

Corollary 1 (Roman-Flores et al. [?]). Let $g : [0, 1] \rightarrow [0, \infty)$ be a non-increasing function and let μ be the Lebesgue measure on $\mathbb{R}$. Then

$$s \left((S) \int_0^1 (1-t)^{s-1} g(t)^s d\mu \right) \geq \left((S) \int_0^1 g(t) d\mu \right)^s$$

holds for all $2 \leq s$.

Corollary 2. Let $g : [0, 1] \rightarrow [0, \infty)$ be a non-increasing function and that μ is the Lebesgue measure on $\mathbb{R}$. Then

$$\begin{aligned} s \left(\frac{(s-1)^{s-1} + s^{s-2}}{2(s-1)^{s-1}} \right) \left((S) \int_0^1 (1-t)^{s-1} g(t)^s d\mu \right) \\ \geq \left((S) \int_0^1 g(t) d\mu \right)^s \end{aligned}$$

holds for all $2 \leq s$.

Example 1. We compare our result with that of Roman-Flores et al. [10] for $s = 2, 3, 4, 5, 10, 20$. That is,

s	$\left((S) \int_0^1 (1-t)^{s-1} d\mu \right)^{-1}$
2	2.000
3	2.618
4	3.148
5	3.630
10	5.692
20	9.110

Our results indicates much lower than that of Roman-Flores et al. [10].

The case of a non-increasing function is similar.

Theorem 4. Let $g : [0, 1] \rightarrow [0, \infty)$ be a non-decreasing

function and let μ be the Lebesgue measure on $\mathbb{R}$. Then

$$\begin{aligned} \left((S) \int_0^1 t^{s-1} d\mu \right)^{-1} \left((S) \int_0^1 t^{s-1} g(t)^s d\mu \right) \\ \geq \left((S) \int_0^1 g(t) d\mu \right)^s \end{aligned}$$

holds for all $1 \geq s$.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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