On Soft w -Structures Defined by Soft Sets

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Abstract

In this work, we introduce the notion of soft w -structure and investigate some basic properties of this new structure by using the concept of soft set. Moreover, we study the notions of soft $w-T_0$ (soft $w-T_1$, soft $w-T_2$).

Keywords: Soft set, Soft topology, Soft w -structure, Soft $w-T_0$ ($w-T_1$, $w-T_2$).

1. Introduction and Preliminaries

In 1999, Molodtsov [1] initiated the notion of soft set theory as a new mathematical tool which is free from the complex problems. Later on Maji et al. [2] proposed several operations on soft sets and some basic properties and then Pei and Miao [3] investigated the relationships between soft sets and information systems.

In 2011, Shabir and Naz [4] introduced the notion of soft topological spaces and the author [5] corrected some their results. Zorlutuna et al. [6] continued to study the properties of soft topological spaces by defining the concepts of interior and soft neighborhoods in soft topological spaces. In 2011, Cagman et al. [7] defined soft topological spaces by modifying the soft set. Also, Roy and Samanta [8] strengthen the definition of the soft topological spaces presented in [7].

In 2017, with the aim of generalizing the notion of soft topology, Zakari et al. [9] introduced a soft weak structure. Recently, Al-Saadi and Min [10] investigated the notion of soft generalized closed sets in a soft weak structure.

Meanwhile, Min and Kim [11] introduced a new notion called weak structures as the following: Let X be a non-empty set and $P(X)$ be its power set. A structure ω on X is called a *weak structure* on X if and only if (i) $\phi \in \omega$: (ii) For $U, V \in \omega$, $U \cap V \in \omega$.

In this work, by applying the notion of weak structure in [11], we want to introduce the new notion of soft w -structure which is a generalized soft topological structure defined by Shabir and Naz [10] and a stronger structure than soft weak structure defined by Zakari et al. [9]. And we investigate some basic properties of this new class by using the notion of soft set. Moreover, we study the notions of soft $w-T_0$ (soft $w-T_1$, soft $w-T_2$) and some properties of such notions. From now on, let X be a non-empty common universe, E a set of parameters, and $P(X)$ denote the power set of X .

Definition 1.1 ([1]). For $A \subseteq E$, a pair (F, A) is called a soft set over X , where F is a mapping given by $F : A \rightarrow P(X)$. For $e \in A$, $F(e)$ may be considered as the set of e -approximate elements of the soft set (F, A) .

Definition 1.2 ([2]). A soft set (F, A) over X is said to be:

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- 1) A null soft set denoted by $\tilde{\emptyset}$ if $F(e) = \emptyset$ for all $e \in A$.
- 2) An absolute soft set denoted by $\tilde{X}$ if $F(e) = X$ for all $e \in A$.

Definition 1.3 ([2]). For any two soft sets (F, A) and (G, B) defined over a common universe X , we have:

- 1) $(F, A) \tilde{\subset} (G, B)$ iff $A \subseteq B$ and $F(e) \subseteq G(e)$ for all $e \in A$.
- 2) $(F, A) = (G, B)$ iff $(F, A) \tilde{\subset} (G, B)$ and $(G, B) \tilde{\subset} (F, A)$.
- 3) $(F, A) \tilde{\cup} (G, B) = (H, C)$ where $C = A \cup B$ and

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cup G(e), & \text{if } e \in A \cap B, \end{cases}$$

for all $e \in C$.

- 4) $(F, A) \tilde{\cap} (G, B) = (K, D)$ where $D = A \cap B$ and $K(e) = F(e) \cap G(e)$ for all $e \in D$.
- 5) $x \in (F, A)$ where $x \in X$ iff $x \in F(e)$ for all $e \in A$ and $x \notin (F, A)$ whenever $x \notin F(e)$ for some $e \in A$.

Definition 1.4 ([12]). For a soft set (F, A) over X , the relative complement of (F, A) (denoted by $(F, A)'$) is defined by $(F, A)' = (F', A)$, where $F' : A \rightarrow P(X)$ is given by $F'(e) = X - F(e)$ for all $e \in A$.

Definition 1.5 ([4]). Let τ be the collection of soft sets over X . Then τ is called a *soft topology* on X if τ satisfies the following axioms:

- 1) $\tilde{\emptyset}, \tilde{X}$ belong to τ .
- 2) The union of any number of soft sets in τ belong to τ .
- 3) The intersection of any two soft sets in τ belong to τ .

The triple (X, τ, E) is called a *soft topological space* over X . The member of τ are said to be *soft open* in X . A soft set (F, E) over X is said to be *soft closed* in X if its relative complement $(F, E)'$ belong to τ .

2. Soft w -Structures

Definition 2.1. Let sw be the collection of soft sets over X . Then sw is called a *soft w -structure* on X if sw satisfies the following axioms:

- $\tilde{\emptyset}, \tilde{X}$ belong to sw .

- The intersection of any two soft sets in sw belongs to sw .

The triple (X, sw, E) is called a *soft w -space* over X . The member of sw is said to be *soft w -open* in X . A soft set (F, E) over X is said to be *soft w -closed* in X if its relative complement $(F, E)'$ belongs to sw .

Remark 2.2. Let sw be a soft w -structure over X . The soft w -structure sw is a kind of generalized soft topology and a stronger structure than a soft weak structure defined by Zakari et al. [9] as the following: Let X be a non-empty set and E a set of parameters. A collection ω of soft sets defined over X with respect to E is called a *soft weak structure* [9] iff $\tilde{\emptyset} \in \omega$.

Example 2.3. Let $X = \{h_1, h_2, h_3, h_4\}$, $E = \{e_1, e_2\}$ and $sw = \{\tilde{\emptyset}, \tilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$, where

$$\begin{aligned} F_1(e_1) &= \{h_2, h_3\}, & F_2(e_2) &= \{h_1, h_2\}; \\ F_2(e_1) &= \{h_1, h_2\}, & F_3(e_2) &= \{h_1, h_3\}; \\ F_3(e_1) &= \{h_2\}, & F_4(e_2) &= \{h_1\}. \end{aligned}$$

Then sw is a soft w -structure over X with respect to E but not a soft topology.

Definition 2.4. Let sw be a soft w -structure over X with respect to E . For a soft set (F, E) over X , the *soft w -closure* of (F, E) (simply, $c_{sw}(F, E)$) and the *soft w -interior* of (F, E) (simply, $i_{sw}(F, E)$) are defined as the following:

- $i_{sw}(F, E) = \tilde{\cup}\{(G, E) : (G, E) \tilde{\subset} (F, E), (G, E) \in sw\}$.
- $c_{sw}(F, E) = \tilde{\cap}\{(H, E) : (F, E) \tilde{\subset} (H, E), (H, E)' \in sw\}$.

Theorem 2.5. Let sw be a soft w -structure over X with respect to the parameters set E and (F, E) a soft set. If there exists a soft w -open set (G, E) such that $x \in (G, E) \tilde{\subset} (F, E)$, then $x \in i_{sw}(F, E)$

Proof. It is obvious. □

Example 2.6. As in Example 2.3, consider the soft w -structure sw over X with respect to E and a soft set (F_4, E) as follows:

$$F_4(e_1) = \{h_1, h_2, h_3\}, \quad F_4(e_2) = \{h_1, h_2, h_3\}.$$

Then $(F_4, E) = i_{sw}(F_4, E)$. For $h_3 \in i_{sw}(F_4, E)$, there is no a soft w -open set containing h_3 in sw . So the converse of Theorem 2.5 is not always true.

Theorem 2.7. Let sw be a soft w -structure over X with respect to the parameters set E and (F, E) a soft set. If $x \in c_{sw}(F, E)$,

then $(G, E) \tilde{\cap} (F, E) \neq \emptyset$ for all $(G, E) \in sw$ such that $x \in (G, E)$.

Proof. Let $x \in c_{sw}(F, E)$. Suppose that there exists an element $(G, E) \in sw$ such that $x \in (G, E)$ and $(F, E) \tilde{\cap} (G, E) = \emptyset$. Then $(F, E) \subset (G, E)'$, so $c_{sw}(F, E) \tilde{\subset} (G, E)'$ and $x \notin c_{sw}(F, E)$. So it is a contradiction. $\square$

Example 2.8. Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $sw = \{\emptyset, \tilde{X}, (F_1, E), (F_2, E), (F_3, E)\}$ where

$$\begin{aligned} F_1(e_1) &= \{h_2, h_3\}, & F_2(e_2) &= \{h_1, h_2\}; \\ F_2(e_1) &= \{h_1, h_2\}, & F_3(e_2) &= \{h_1, h_3\}; \\ F_3(e_1) &= \{h_2\}, & F_4(e_2) &= \{h_1\}. \end{aligned}$$

Then sw is a soft w -structure over X with respect to E . Consider a soft set (F_4, E) defined as:

$$F_4(e_1) = \{h_1\}, \quad F_4(e_2) = \{h_3\}.$$

Since (F_4, E) is soft w -closed, $(F_4, E) = c_{sw}(F_4, E)$. For $h_1 \in X$, (F_2, E) is the only soft w -open set and $(F_4, E) \tilde{\cap} (F_2, E) \neq \emptyset$, however, $h_1 \notin c_{sw}(F_4, E)$. So the converse of Theorem 2.7 is not always true.

Theorem 2.9. Let sw be a soft w -structure defined over X with respect to the parameters set E and (F, E) be a soft set.

- If (F, E) is a soft w -open set, then $(F, E) = i_{sw}(F, E)$.
- If (F, E) is a soft w -closed set, then $(F, E) = c_{sw}(F, E)$.

Proof. From the definitions of soft w -interior and soft w -closure, it is obvious. $\square$

But the converses in Theorem 2.9 are not always true as shown the next example.

Example 2.10. Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $sw = \{\emptyset, \tilde{X}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E)\}$, where

$$\begin{aligned} F_1(e_1) &= \{h_3\}, & F_1(e_2) &= \{h_2\}; \\ F_2(e_1) &= \{h_2, h_3\}, & F_2(e_2) &= \{h_1, h_2\}; \\ F_3(e_1) &= \{h_1, h_2\}, & F_3(e_2) &= \{h_1, h_3\}; \\ F_4(e_1) &= \{h_1\}, & F_4(e_2) &= \{h_3\}; \\ F_5(e_1) &= \{h_2\}, & F_4(e_2) &= \{h_1\}; \\ F_6(e_1) &= \{h_1, h_3\}, & F_4(e_2) &= \{h_2, h_3\}; \end{aligned}$$

Then sw is a soft w -structure over X with respect to E . For a soft set (F_5, E) , $c_{sw}(F_5, E) = (F_5, E)$ but (F_5, E) is not soft w -closed. And, for a soft set (F_6, E) , $i_{sw}(F_6, E) = (F_6, E)$ but (F_6, E) is not soft w -open.

Theorem 2.11. Let sw be a soft w -structure over X with respect to E . Let (F, E) and (G, E) be two soft sets over X . Then:

- $i_{sw}(F, E) \tilde{\subset} (F, E)$.
- If $(F, E) \tilde{\subset} (G, E)$, then $i_{sw}(F, E) \tilde{\subset} i_{sw}(G, E)$.
- $i_{sw}((F, E) \tilde{\cap} (G, E)) = i_{sw}(F, E) \tilde{\cap} i_{sw}(G, E)$.
- $i_{sw}(i_{sw}(F, E)) = i_{sw}(F, E)$.

Proof. (1) and (2) are obvious.

(3) It is obvious that $i_{sw}((F, E) \tilde{\cap} (G, E)) \tilde{\subset} i_{sw}(F, E) \tilde{\cap} i_{sw}(G, E)$ from (2). For soft w -open sets $(U, E) \tilde{\subset} (F, E)$ and $(V, E) \tilde{\subset} (G, E)$, $(U, E) \tilde{\cap} (V, E)$ is a soft w -open set contained in $(F, E) \tilde{\cap} (G, E)$. This implies that

$$i_{sw}(F, E) \tilde{\cap} i_{sw}(G, E) \tilde{\subset} i_{sw}((F, E) \tilde{\cap} (G, E)).$$

(4) From (1), it follows $i_{sw}(i_{sw}(F, E)) \tilde{\subset} i_{sw}(F, E)$. For any soft w -open set (U, E) such that $(U, E) \tilde{\subset} i_{sw}(F, E)$, $(U, E) = i_{sw}(U, E) \tilde{\subset} i_{sw}(i_{sw}(F, E))$, and so $i_{sw}(F, E) \tilde{\subset} i_{sw}(i_{sw}(F, E))$. Consequently, we have $i_{sw}(i_{sw}(F, E)) = i_{sw}(F, E)$. $\square$

Theorem 2.12. Let sw be a soft w -structure defined over X with respect to E . If (F, E) and (G, E) are two soft sets over X , then:

- $(F, E) \tilde{\subset} c_{sw}(F, E)$.
- If $(F, E) \tilde{\subset} (G, E)$, then $c_{sw}(F, E) \tilde{\subset} c_{sw}(G, E)$.
- $c_{sw}(F, E) \tilde{\cup} c_{sw}(G, E) = c_{sw}((F, E) \tilde{\cup} (G, E))$.
- $c_{sw}(c_{sw}(F, E)) = c_{sw}(G, E)$.
- $i_{sw}(F, E)' = (c_{sw}(F, E))'$ and $c_{sw}(F, E)' = (i_{sw}(F, E))'$.

Proof. It is similar to the proof of Theorem 2.11. $\square$

Now, we introduce the separation axioms in soft w -space with a soft w -structure sw .

Definition 2.13. Let sw be a soft w -structure over X with respect to E . A soft w -space (X, sw, E) is called:

- $w - T_0$ if for each $x, y \in X$ such that $x \neq y$, there exists a soft w -open set (F, E) such that $x \in (F, E)$ and $y \notin (F, E)$ or $x \notin (F, E)$ and $y \in (F, E)$.
- $w - T_1$ if for each $x, y \in X$ such that $x \neq y$, there exist soft w -open sets (F, E) and (G, E) such that $x \in (F, E)$ and $y \notin (F, E)$ and $x \notin (G, E)$ and $y \in (G, E)$.
- $w - T_2$ if for each $x, y \in X$ such that $x \neq y$, there exist soft w -open sets (F, E) and (G, E) such that $x \in (F, E)$, $y \in (G, E)$ and $(F, E) \tilde{\cap} (G, E) = \tilde{\emptyset}$.

We have the following diagram:

$$\text{soft } w - T_2 \Rightarrow \text{soft } w - T_1 \Rightarrow \text{soft } w - T_0.$$

Example 2.14. Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $sw = \{\tilde{\emptyset}, \tilde{X}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E), (F_6, E)\}$, where

$$\begin{aligned} F_1(e_1) &= \{h_3\}, & F_1(e_2) &= \{h_3\}; \\ F_2(e_1) &= \{h_1, h_2\}, & F_3(e_2) &= \{h_1, h_3\}; \\ F_3(e_1) &= \{h_2, h_3\}, & F_2(e_2) &= \{h_1, h_2\}; \\ F_4(e_1) &= \{h_2\}, & F_4(e_2) &= \{h_1\}; \\ F_5(e_1) &= \{h_3\}, & F_4(e_2) &= \emptyset; \\ F_6(e_1) &= \emptyset, & F_4(e_2) &= \{h_3\}. \end{aligned}$$

Then sw is a soft w -structure over X with respect to E . It is obviously a soft $w - T_1$ space. For $h_1, h_2 \in X$, (F_2, E) and (F_3, E) are unique soft w -open sets of h_1, h_2 , respectively. But $(F_2, E) \tilde{\cap} (F_3, E) \neq \tilde{\emptyset}$. So (X, sw, E) is not soft $w - T_2$.

Example 2.15. Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $sw = \{\tilde{\emptyset}, \tilde{X}, (F_1, E), (F_2, E), (F_3, E), (F_4, E)\}$, where

$$\begin{aligned} F_1(e_1) &= \{h_1\}, & F_1(e_2) &= \{h_1\}; \\ F_2(e_1) &= \{h_2\}, & F_3(e_2) &= \{h_2\}; \\ F_3(e_1) &= \{h_1, h_3\}, & F_4(e_2) &= X; \\ F_4(e_1) &= \emptyset, & F_4(e_2) &= \{h_2\}. \end{aligned}$$

Then sw is a soft w -structure over X with respect to E . It is obviously a soft $w - T_0$ space but it is not soft $w - T_1$.

Let sw be a soft w -structure over X with respect to E . A soft w -space (X, sw, E) is called *relative soft $w - T_0$* if for each $x, y \in X$ such that $x \neq y$, there exists a soft w -open set (F, E) such that $x \in (F, E)$ and $y \in (F, E)'$ or $x \in (F, E)'$ and $y \in (F, E)$.

Theorem 2.16. Let sw be a soft w -structure on X . If X is a relative soft $w - T_0$ space, then for each $x, y \in X$ such that $x \neq y$, we have $c_{sw}(x, E) \neq c_{sw}(y, E)$.

Proof. Let X be a relative soft $w - T_0$ and $x, y \in X$ such that $x \neq y$. Then there exists a soft w -open set (F, E) such that $x \in (F, E)$ and $y \in (F, E)'$. Therefore $(F, E)'$ is a soft w -closed set such that $x \notin (F, E)'$ and $y \in (F, E)'$. Since $c_{sw}(y, E)$ is the intersection of all soft w -closed subsets containing (y, E) , $c_{sw}(y, E) \subset (F, E)'$ and hence $x \notin c_{sw}(y, E)$. Thus $c_{sw}(x, E) \neq c_{sw}(y, E)$. $\square$

Theorem 2.17. Let sw be a soft w -structure on X . If $y \in c_{sw}(x, E)$, then for each soft w -open set (G, E) containing y , there exists a parameter $e \in E$ such that $x \in G(e)$.

Proof. Let $y \in (c_{sw}(x, E))$. Then by Theorem 2.7, $(G, E) \tilde{\cap} (x, E) \neq \tilde{\emptyset}$ for all $(G, E) \in sw$ such that $y \in (G, E)$. Since $(G, E) \tilde{\cap} (x, E) \neq \tilde{\emptyset}$, there exists a parameter $e \in E$ such that $x \in G(e)$. $\square$

Theorem 2.18. Let sw be a soft weak structure on X . A soft w -space (X, sw, E) is soft $w - T_1$ if (x, E) is soft w -closed set for all $x \in X$.

Proof. Let $x, y \in X$ such that $x \neq y$. Then $(x, E)'$ and $(y, E)'$ are soft w -open sets such $y \in (x, E)'$, $x \in (x, E)'$ and $y \notin (y, E)'$, $x \in (y, E)'$. Hence X is soft $w - T_1$. $\square$

3. Conclusions

The author introduced the notion of soft w -structure and investigated some basic properties of this new structure. In the next research, the author will introduce the associated soft w -structures induced by soft topologies and study the relationship between soft w -structures and associated soft w -structure induced by soft topologies.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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