

Order ν Entropy and Cross Entropy of Uncertain Variables for Portfolio Selection

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Abstract

In this study, we proposed definition of order ν entropy and order ν cross entropy of uncertain variables under uncertainty theory. Moreover, order ν entropy and order ν cross entropy of uncertain variables were applied to mean-variance portfolio selection model. We also attempted to examine the applications of these measures with different order ν values. The effect of the ν in order ν entropy and cross entropy on portfolio selection were considered using the order ν entropy-mean-variance and 571781799order ν entropy-mean-variance presented models. As a result of this approach, by using different values of ν , diversity of asset allocations could be achieved.

Keywords: Uncertain variable, Order ν entropy, Order ν cross entropy, Portfolio selection

1. Introduction

The purpose of portfolio selection is to allocate optimal assets to purchase stocks. In general, to achieve this goal, returns are considered as random variables. The uncertainty variables are used as returns in the absence of historical data. Accordingly, the use of entropy in portfolio selection is considered for measuring the uncertainty of the uncertain variables. As an application of this concept, we can consider the generalized entropy maximization model and cross entropy minimization model under the uncertainty environment.

Portfolio selection is a very extensive topic in finance, which has been introduced to this field in the 1950's, and the interest in it does not seem to be subsiding. Markowitz [1] was the first scientist to introduce the modern portfolio selection theory. Later, many researchers revised or developed the model with new methods or elements to improve the results. A typical extension was suggested by Philippatos and Wilson [2], who employed entropy to measure the risk of the portfolio selection. After that, several important works were published in this topic, for example see Zhou et al. [3] and Yu et al. [4].

Fuzzy entropy is a way to characterize the uncertainty on the possible values of fuzzy variables, which has been studied by many researchers such as Bhandari and Pal [5], De Luca and Termini [6], and Liu [7]. Within the framework of credibility theory, Li and Liu [8] presented an entropy for fuzzy variable. Li and Liu [9] proposed the maximum entropy. Based on the concept of fuzzy entropy, Li et al. [10] proposed the maximum optimization model by minimizing the uncertainty of the fuzzy objective under certain expected constraints. Further, Qin et al. [11] established credibilistic cross-entropy minimization models for portfolio optimization with fuzzy returns in the framework of credibility theory. Bhattacharyya et al.

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[12] introduced the cross-entropy, mean, variance, skewness model. Cross entropy was used to quantify the level of dispersion for the fuzzy return. Yari et al. [13] presented the Renyi entropy-mean-variance maximization and Renyi cross entropy-mean-variance minimization models for portfolio selection with fuzzy return under the credibility theory framework. Kar et al. [14] introduced a fuzzy bi-objectives portfolio model with objectives “fuzzy VaR ratio” and “fuzzy Sharpe ratio”. They tested the performance of the model with different evolutionary algorithms.

In the situations where historical data is not available, another feasible way is to estimate returns using expert opinion based on their subjective evaluation under the uncertainty theory (Liu [15]). Recently, portfolio selection problems have been studied under uncertainty conditions. By using the uncertainty theory, several researchers, including Qin et al. [11], who formulated the uncertain counterpart of mean-variance model, Liu and Qin [16], Huang and Qiao [17], Yao and Ji [18], took the security returns as uncertain variables. In order to measure the uncertainty of a variable, entropy was provided by Liu [19] under the uncertainty theory. Subsequently, some properties of entropy for uncertain variables were investigated by Dai and Chen [20]. In order to address the divergence of uncertain variables via uncertain distribution, Chen et al. [21] proposed cross entropy of uncertain variables. Bhattacharyya et al. [22] showed that their developed mean-entropy-skewness model in an uncertain environment is more effective in comparison with other earlier proposed models.

This paper focuses on the portfolio optimization with uncertain returns subject to expert evaluation. Section 2 reviews some important concepts and measures of uncertain variables and explain their implications in portfolio selection. Section 3 proposes order v entropy and order v cross entropy for uncertain variables. In Section 4, we establish order v entropy maximization and order v cross entropy minimization models for portfolio optimization with uncertain returns and give numerical examples. We demonstrate that using different values of v can lead to a higher diversification of the asset allocations. Finally, some conclusions are listed in Section 5.

2. Preliminary

The uncertainty theory, introduced by Liu [15] is a branch of mathematics that studies the behavior of human uncertainty. In this section, we review some basic concepts about uncertain measures and uncertain variables, which are related with this

paper.

Let Γ be a nonempty set and \mathfrak{S} be a σ -algebra over Γ . Each element $\Lambda \in \mathfrak{S}$ is called an event. In order to indicate the chance that Λ will happen, Liu [23] proposed the following four axioms to ensure that $\mathcal{M}\{\Lambda\}$ satisfying certain mathematical properties.

Axiom 1 (Normality). $\mathcal{M}\{\Lambda\} = 1$ for the universal set Γ .

Axiom 2 (Self-duality). $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ .

Axiom 3 (Countable subadditivity). for every countable sequence of events $\Lambda_1, \Lambda_2, \Lambda_3, \dots$, we have $\mathcal{M}\left\{\sum_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$. Then the triple $(\Gamma, \mathfrak{S}, \mathcal{M})$ will be called an uncertain space.

Axiom 4 (Product). Let $(\Gamma_i, \mathfrak{S}_i, \mathcal{M}_i)$ be the uncertain space for $i = 1, 2, \dots$, then the product uncertain measure \mathcal{M} is an uncertain measure satisfying $\mathcal{M}\left\{\prod_{i=1}^{\infty} \Lambda_i\right\} = \prod_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$, where Λ_i is the arbitrarily chosen event from \mathfrak{S}_i for every $i = 1, 2, \dots$

If ξ is an uncertain variable, then its uncertainty distribution is define as follows:

$$\Phi(x) = \mathcal{M}(\xi \leq x), \quad (1)$$

where $x \in R$.

Definition 2.1. In uncertainty theory, Liu and Liu [24] defined the expected value and variance of ξ as follows:

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\} dr - \int_{-\infty}^0 \mathcal{M}\{\xi \leq r\} dr. \quad (2)$$

Provided that at least one of the two integrals is finite.

$$V(\xi) = E\left[(\xi - E[\xi])^2\right]. \quad (3)$$

If $\xi = Z(a, b, c)$ is a zigzag uncertain variable with following uncertainty distribution

$$\Phi(r) = \begin{cases} 0, & \text{if } r \leq a, \\ (r - a)/2(b - a), & \text{if } a \leq r \leq b, \\ (r + c - 2b)/2(c - b), & \text{if } b \leq r \leq c, \\ 1, & \text{if } r \geq c, \end{cases} \quad (4)$$

where a, b, c are real numbers with $a < b < c$. Further, the zigzag uncertain variable has an expected value $E[\xi] = \frac{a+2b+c}{4}$

and variance

$$\text{var} [\xi] = \begin{cases} \frac{\left(\frac{33(b-a)^3 + 21(b-a)^2(c-b)}{384(b-a)} \right)}{\text{if } b-a > c-b,} \\ \frac{\left(\frac{33(c-b)^3 + 21(c-b)^2(b-a)}{384(c-b)} \right)}{\text{if } b-a < c-b.} \end{cases}$$

Definition 2.2. Let ξ be a uncertain variable with uncertain distribution Φ . Li and Liu [8] presented the following definition for uncertain entropy

$$H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x))dx, \quad (5)$$

where $S(t) = -t \ln t - (1-t) \ln (1-t)$.

Definition 2.3 (Chen et al. [21]). Let ξ and η be uncertain variables. Then the cross entropy of ξ from η is defined as

$$D[\xi; \eta] = \int_{-\alpha}^{+\alpha} T(\mathcal{M}\{\xi \leq x; \mathcal{M}\{\eta \leq x\})dx, \quad (6)$$

where $T(s; t) = s \ln(\frac{s}{t}) + (1-s) \ln(\frac{(1-s)}{(1-t)})$. In terms of distribution function cross entropy is defined as

$$D[\xi; \eta] = \int_{-\alpha}^{+\alpha} (\Phi_{\xi}(x) \ln \left(\frac{\Phi_{\xi}(x)}{\Phi_{\eta}(x)} \right) + (1 - \Phi_{\xi}(x)) \ln \left(\frac{1 - \Phi_{\xi}(x)}{1 - \Phi_{\eta}(x)} \right)) dx, \quad (7)$$

where Φ_{ξ} and Φ_{η} are the respective distribution functions of uncertain variables ξ and η .

3. The Order v Entropy and Cross Entropy for Uncertain Sets

The entropy concept was introduced by Shanon [25] and Renyi [26] defined the order v entropy of a probability distribution (p_1, p_2, \dots, p_n) as $\frac{1}{1-v} \ln(\sum_{i=1}^n p_i^v)$, $v > 0$. In order to measure the uncertainty of uncertain variable, entropy was provided by Liu [19] in uncertainty theory. We introduce a definition of order v fuzzy entropy as follows:

Definition 3.1. Suppose that ξ is an uncertain variable with

uncertain distribution Φ . Then its order v entropy is defined by

$$H[\xi] = \int_{-\infty}^{+\infty} S(\mathcal{M}(\xi \leq x))dx, \quad (8)$$

where $S(t) = \frac{1}{1-v} \ln[t^v + (1-t)^v]$, $v > 0$, $v \neq 0$.

The order v cross-entropy of ξ and η can be written as:

$$H[\xi] = \frac{1}{1-v} \int_{-\infty}^{+\infty} \log(\Phi^v(x) + (1-\Phi(x))^v) dx. \quad (9)$$

Theorem 1. Let ξ is zigzag uncertain variable with uncertainty distribution $\xi = Z(a, b, c)$. Then the order v entropy of ξ is

$$H[\xi] = \frac{2(b-a)}{1-v} \int_0^{1/2} \log(x^v + (1-x)^v) dx + \frac{2(c-b)}{1-v} \int_{1/2}^1 \log(x^v + (1-x)^v) dx. \quad (10)$$

Proof. Considering Eqs. (5) and (9), we obtain the following:

$$H[\xi] = \frac{1}{1-v} \left[\int_a^b \log \left(\left(\frac{x-a}{2(b-a)} \right)^v + \left(1 - \frac{x-a}{2(b-a)} \right)^v \right) dx + \int_b^c \log \left(\left(\frac{x+c-2b}{2(c-b)} \right)^v + \left(1 - \frac{x+c-2b}{2(c-b)} \right)^v \right) dx \right]$$

By the *changes of variable* technique, Theorem 1 can be easily proved. Therefore, the theorem is proved, and solving this integral numerically for different values of v , ($v > 0$) is possible. \square

Cross entropy was first proposed by Kullback and Leibler [27] to measure the difference between two probability distribution. Renyi [26] defined the order v cross entropy of a probability distribution (p_1, p_2, \dots, p_n) as $D_R(P||Q) = \frac{1}{v-1} \log \left(\sum_{i=1}^n p_i^v q_i^{1-v} \right)$, $v \neq 1$, $v > 0$. Bahandari and Pal [5] introduced order v cross entropy based on fuzzy theory, Chen et al. [21] introduced cross entropy for uncertain variables. We define order v cross entropy for uncertain sets as follows:

Definition 3.2. Let ξ and η be two uncertain variables. Then, the order v cross entropy of ξ and η is defined as

$$D[\xi; \eta] = \int_{-\alpha}^{+\alpha} T(\mathcal{M}\{\xi \leq x; \mathcal{M}\{\eta \leq x\})dx, \quad (11)$$

where $T(s; t) = \frac{1}{v-1} \log [t^v s^{1-v} + (1-t)^v (1-s)^{1-v}]$, $v > 0$, $v \neq 1$.

It is easy to verify that $T(s, t)$ is strictly convex with respect to (t, s) and attains its minimum value 0 on the line $s = t$, also for any $0 \leq s \leq 1$ and $0 \leq t \leq 1$, we have $T(s, t) = T(1-s, 1-t)$.

The order v cross-entropy of ξ and η can be written as

$$D[\xi; \eta] = \int_{-\infty}^{+\infty} \frac{1}{v-1} \log [\Phi_{\xi}^v(x) \Phi_{\eta}^{1-v}(x) + (1 - \Phi_{\xi}(x))^v (1 - \Phi_{\eta}(x))^{1-v}] dx. \quad (12)$$

Theorem 2. Let ξ and η be zigzag uncertain variables with uncertainty distribution $\xi = Z(a, b, c)$ and $\eta = Z(d, b, e)$, ($d \leq a < b < c \leq e$), respectively. Then the order v cross-entropy of ξ and η is

$$\begin{aligned} D[\xi; \eta] = & \left[- \int_d^a \log \left(\frac{2b-d-x}{2(b-d)} \right) dx - \int_c^e \log \left(\frac{x+e-2b}{2(e-b)} \right) dx \right. \\ & + \frac{1}{v-1} \int_a^b \log \left(\left(\frac{(x-a)(b-d)}{(x-d)(b-a)} \right)^v \left(\frac{x-d}{2(b-d)} \right) \right. \\ & + \left. \left(\frac{(2b-a-x)(b-d)}{(2b-d-x)(b-a)} \right)^v \left(\frac{2b-d-x}{2(b-d)} \right) \right) dx \\ & + \frac{1}{v-1} \int_b^c \log \left(\left(\frac{(x+c-2b)(e-b)}{(x+e-2b)(c-b)} \right)^v \left(\frac{x+e-2b}{2(e-b)} \right) \right. \\ & \left. \left. + \left(\frac{(c-x)(e-b)}{(e-x)(c-b)} \right)^v \left(\frac{(e-x)}{2(e-b)} \right) \right) dx \right]. \quad (13) \end{aligned}$$

Proof. Considering Eqs. (6) and (12), we obtain the following:

$$\begin{aligned} D[\xi; \eta] = & \frac{1}{v-1} \left[\int_d^a \log \left(1 - \frac{x-d}{2(b-d)} \right)^{1-v} dx \right. \\ & + \int_a^b \log \left(\left(\frac{x-a}{2(b-a)} \right)^v \left(\frac{x-d}{2(b-d)} \right)^{1-v} \right. \\ & + \left. \left(1 - \frac{x-a}{2(b-a)} \right)^v \left(1 - \frac{x-d}{2(b-d)} \right)^{1-v} \right) dx \\ & + \int_b^c \log \left(\left(\frac{x+c-2b}{2(c-b)} \right)^v \left(\frac{x+e-2b}{2(e-b)} \right)^{1-v} \right. \\ & \left. + \left(1 - \frac{x+c-2b}{2(c-b)} \right)^v \left(1 - \frac{x+e-2b}{2(e-b)} \right)^{1-v} \right) dx \end{aligned}$$

$$+ \int_c^e \log \left(\frac{x+e-2b}{2(e-b)} \right)^{1-v} dx \Big].$$

In particular if the uncertainty distributions of ξ and η are $Z(2, 3, 4)$ and $Z(1, 3, 5)$, respectively, using the theorem above, we get $D[\xi; \eta] = 0.14$, when $v \rightarrow 2$. \square

4. The Order v Entropy Maximization and Cross Entropy Minimization Models

In this section, Kapur and Kesavan [28] entropy maximization and cross-entropy minimization model is extended to the portfolio optimization with uncertain returns. Let ξ_i be the i th return of the security, and x_i is the proportion of capital allocated for the i -th security, where $i = 1, 2, \dots, n$. Let $\xi_1, \xi_2, \dots, \xi_n$ be the uncertain variables in the uncertain space $(\Gamma, \mathfrak{F}, \mathcal{M})$. Then, the total return from the investment is $\xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n$, which is an uncertain variable. Then, order v entropy-mean-variance model is presented as follows:

$$\begin{cases} \max_{x_i} H[\xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n] \\ \text{subject to:} \\ E[\xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n] \geq r_0, \\ V[\xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n] \leq d_0, \\ x_1 + x_2 + \dots + x_n = 1, \quad x_i \geq 0, \quad i = 1, 2, \dots, n, \end{cases} \quad (14)$$

where r_0 is the predetermined expected return and d_0 is the predetermined risk for the portfolio.

Theorem 3. Suppose that each security return is a zigzag variable denoted by $\xi_i = Z(a_i, b_i, c_i)$ ($i = 1, 2, \dots, n$). Then the model (14) can be transformed into the following crisp form:

$$\begin{cases} \max \frac{\sum_{i=1}^n 2(c_i - a_i)x_i}{1-v} \int_0^1 \log(x_i^v + (1-x_i)^v) dx_i \\ \text{subject to:} \\ \sum_{i=1}^n (a_i + 2b_i + c_i)x_i \geq 4r_0, \\ 11 \left(\sum_{i=1}^n x_i(c_i - a_i)^2 \left| \sum_{i=1}^n x_i(2b_i - a_i - c_i) \right| \right. \\ \left. + 2 \left(8 \sum_{i=1}^n x_i(c_i - a_i) + 3 \left| \sum_{i=1}^n x_i(2b_i - a_i - c_i) \right| \right) \right. \\ \left. \times \left(\sum_{i=1}^n x_i(c_i - b_i)^2 + \left(\sum_{i=1}^n x_i(b_i - a_i)^2 \right) \right) \right) \end{cases}$$

$$\left\{ \begin{array}{l} \leq 192d_0 \left(\sum_{i=1}^n x_i(c_i - a_i) + \left| \sum_{i=1}^n x_i(2b_i - c_i - a_i) \right| \right), \\ x_1 + x_2 + \dots + x_n = 1, \quad x_i \geq 0, \quad i = 1, 2, \dots, n. \end{array} \right.$$

Proof. Note that in uncertain environment, $E[\xi_1 x_1 + \xi_2 x_2 + \dots + x_n \xi_n] \neq x_1 E[\xi_1] + x_2 E[\xi_2] + \dots + x_n E[\xi_n]$ for uncertain variables $\xi_1, \xi_2, \dots, \xi_n$. However, the inequality will become equality when $\xi_1, \xi_2, \dots, \xi_n$ are independent. Further, we assume that security returns are all zigzag uncertain variables, denoting the return of security i by $\xi_i = Z(a_i, b_i, c_i)$. It follows that the portfolio return $\sum_{i=1}^n \xi_i x_i = Z(\sum_{i=1}^n a_i x_i, \sum_{i=1}^n b_i x_i, \sum_{i=1}^n c_i x_i)$ is also a zigzag uncertain variable.

Since the cross entropy is a common method for measuring the degree of divergence of uncertain variables, we formulate different cross entropy minimization model for portfolio optimization. Suppose that η is a prior uncertain investment return for an investor. Then, the mean-variance-order v cross entropy model is presented as follows:

$$\left\{ \begin{array}{l} \min_{x_i} D[\xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n; \eta] \\ \text{subject to:} \\ E[\xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n] \geq r_0, \\ V[\xi_1 x_1 + \xi_2 x_2 + \dots + \xi_n x_n] \leq d_0, \\ x_1 + x_2 + \dots + x_n = 1, \quad x_i \geq 0, \quad i = 1, 2, \dots, n. \end{array} \right. \quad (15)$$

Theorem 4. Suppose that each security return is a zigzag variable denoted by $\xi_i = Z(a_i, b, c_i)$ ($i = 1, 2, \dots, n$). Let the prior investment return be $\eta = (d, b_i, e)$. Then model (15) is equivalent to crisp model,

$$\left\{ \begin{array}{l} \min \frac{v}{v-1} \sum_{i=1}^n (x_i \alpha_i \log \frac{b_i - d}{x_i \alpha_i} + x_i \beta_i \log \frac{c_i - b_i}{x_i \beta_i}) \\ \quad - \frac{1}{v-1} \sum_{i=1}^n (x_i \alpha_i \log 2(b_i - d) + x_i \beta_i \log 2(e - b_i)), \\ \quad \frac{1}{v-1} \int_{a_i}^b \log \left[\left(\frac{x_i - a_i}{x_i - d} \right)^v (x_i - d) \right. \\ \quad \left. + \left(\frac{2b_i - a_i - x_i}{2b_i - d - x_i} \right)^v (2b_i - a_i - x_i) \right] dx_i \\ \quad - \int_d^{a_i} \log(2b_i - d - x_i) dx_i, \\ \quad \frac{1}{v-1} \int_{c_i}^b \log \left[\left(\frac{x_i + c_i - 2b_i}{x_i + e - 2b_i} \right)^v (x_i + e - 2b_i) \right. \end{array} \right.$$

$$\left\{ \begin{array}{l} + \left(\frac{c_i - x_i}{e - x_i} \right)^v (e - x_i) \left] dx_i - \int_{c_i}^e \log(x_i + e - 2b_i) dx_i, \right. \\ \text{subject to:} \\ \sum_{i=1}^n (a_i + 2b_i + c_i) x_i \geq 4r_0, \\ 11 \left(\sum_{i=1}^n x_i(c_i - a_i)^2 \left| \sum_{i=1}^b x_i(2b_i - a_i - c_i) \right| \right. \\ \quad \left. + 2 \left(8 \sum_{i=1}^n x_i(c_i - a_i) + 3 \left| \sum_{i=1}^n x_i(2b_i - a_i - c_i) \right| \right) \right. \\ \quad \left. \times \left(\sum_{i=1}^n x_i(c_i - b_i)^2 + \left(\sum_{i=1}^n x_i(b_i - a_i) \right)^2 \right) \right. \\ \quad \left. \leq 192d_0 \left(\sum_{i=1}^n x_i(c_i - a_i) + \left| \sum_{i=1}^n x_i(2b_i - c_i - a_i) \right| \right), \right. \\ x_1 + x_2 + \dots + x_n = 1, \quad x_i \geq 0, \quad i = 1, 2, \dots, n. \end{array} \right.$$

Proof. The theorem can be easily proved taking into account the relations used in proving Theorem 3. Solving the integral using numeric methods is possible for different values of v . \square

In the rest of this section, minimization order v cross entropy-mean-variance models with 3 different v are applied to the data from Qin and Yao [29], who had used them to illustrate the application of uncertain mean-lower partial moment model. Assume that an investor plans to invest his fund among securities. Further, all the future returns of the securities are assumed to be zigzag uncertain variable. We apply model (15) to determine the optimal portfolio and employ the function “fmincon” in MATLAB (2013a) to solve it. For given minimal return level r_0 and risk level d_0 , we obtain a series of optimal investment strategies for three different v in Tables 1-3, respectively.

Tables 1-3 are obtained with setting the v level at 0.5, 1, and 1.5, respectively. Selection using the order $v = 0.5$ cross entropy-mean-variance leads to a more diverse and decentralized portfolio for a limited amount of assets in comparison to

Table 1. Investment proportion of 5 securities (%) with $v = 0.5$ and prior investment return $\eta_i = (-0.35, b_i, 1.2)$

No.	Return	Allocation
ξ_1	(-0.2, 0.5, 0.9)	$x_1 = 0.29, x_2 = 0.39, x_3 = 0.03,$ $x_4 = 0.05, x_5 = 0.24$
ξ_2	(-0.3, 0.6, 1.0)	
ξ_3	(-0.1, 0.3, 0.8)	
ξ_4	(-0.2, 0.3, 1.0)	
ξ_5	(-0.3, 0.5, 0.7)	

Table 2. Investment proportion of 5 securities (%) with $v = 1$ and prior investment return $\eta_i = (-0.35, b_i, 1.2)$

No.	Return	Allocation
ξ_1	(-0.2,0.5,0.9)	$x_1 = 0.32, x_2 = 0.41, x_3 = 0.00,$ $x_4 = 0.04, x_5 = 0.23$
ξ_2	(-0.3,0.6,1.0)	
ξ_3	(-0.1,0.3,0.8)	
ξ_4	(-0.2,0.3,1.0)	
ξ_5	(-0.3,0.5,0.7)	

Table 3. Investment proportion of 5 securities (%) with $v = 1.5$ and prior investment return $\eta_i = (-0.35, b_i, 1.2)$

No.	Return	Allocation
ξ_1	(-0.2,0.5,0.9)	$x_1 = 0.34, x_2 = 0.46, x_3 = 0.00,$ $x_4 = 0.00, x_5 = 0.2$
ξ_2	(-0.3,0.6,1.0)	
ξ_3	(-0.1,0.3,0.8)	
ξ_4	(-0.2,0.3,1.0)	
ξ_5	(-0.3,0.5,0.7)	

the order ($v = 1, 1.5$) entropy-mean-variance models. Furthermore, solving the model (14) with the returns used in the previous example led to the same investment allocation as calculated in model (15), v being at 0.5, 1, and 1.5.

5. Conclusions

In the present study, we compared the applicability of two models, order v entropy-mean-variance and order v cross entropy-mean-variance, for portfolio selection under the uncertainty set. We showed that there is no difference between the two models in portfolio optimization. It was also presented that lowering the values of the v parameter in the order v cross entropy-mean-variance minimization results in more diversified portfolio selection. We may also conclude that using different values of v in proposed models for portfolio optimization would affect the decision of an investor to allocate his capital to purchase various securities.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

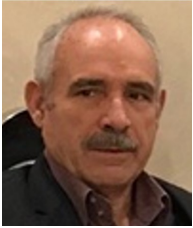
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