Soft ω^* -Paracompactness in Soft Topological Spaces

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Abstract

In this study, we introduce a new concept in soft topological spaces, namely, soft ω^* -paracompactness, and we provide characterizations thereof. Its connection with other related concepts is also studied. In particular, we show that soft ω^* -paracompactness and soft paracompactness are independent of each other. In addition, we study the soft ω^* -paracompactness of the soft topological space generated by an indexed family of ω^* -paracompact topological spaces.

Keywords: ω -open sets, ω^* -paracompact, Soft paracompact, Generated soft topology

1. Introduction

Throughout this paper, we use the notions and terminology in [1] and [2]; moreover, TS and STS stand for “topological space” and “soft topological space,” respectively. Recently, classical methods have been applied to several problems in various fields, such as engineering, social sciences, and medical sciences. Soft sets were defined by Molodtsov [3], and they have found numerous applications. Let X be a universal set and A be a set of parameters. A soft set over X relative to A is a function $G : A \rightarrow \mathcal{P}(X)$. The family of all soft sets over X relative to A is denoted by $SS(X, A)$. If $G \in SS(X, A)$ is such that $G(a)$ is a countable set for all $a \in A$, then G is called a countable soft set. The family of all members of $SS(X, A)$ that are countable is denoted by $CSS(X, A)$. We denote the null soft set and the absolute soft set as 0_A and 1_A , respectively. Soft topological spaces were defined in [4] as follows: A triplet (X, τ, A) , where $\tau \subseteq SS(X, A)$, is called an STS if 0_A and $1_A \in \tau$, and τ is closed under finite soft intersections and arbitrary soft unions. For an STS (X, τ, A) , the members of τ are called soft open sets, and their soft complements are called soft closed sets. Several topological concepts have been extended to the context of STSs [1, 2, 5–28]. The concept of ω -open set was introduced in [29] as a weaker form of an open set as follows: Let $(X, \mathfrak{S})$ be a TS and let $U \subseteq X$; then, U is called ω -open if for every $x \in U$, there is an open set V and a countable subset $C \subseteq X$ such that $x \in V - C \subseteq U$.

The complement of an ω -open set in $(X, \mathfrak{S})$ is called an ω -closed set. The family of all ω -open sets in $(X, \mathfrak{S})$ is denoted by $\mathfrak{S}_\omega$. It is known that $\mathfrak{S}_\omega$ forms a topology on X that contains $\mathfrak{S}$. The study of ω -open sets remains an area that attracts considerable attention [30–35]. Using ω -open sets, the concept of ω^* -paracompactness was defined and studied in ordinary TSs [36]. In [2], the concept of ω -openness was extended to include STSs as follows: Let (X, τ, A) be an STS, and let $G \in SS(X, A)$; then, G is called a soft ω -open set if for all

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$a_x \tilde{\in} H$, there exist $F \in \tau$ and $H \in CSS(X, A)$ such that $a_x \tilde{\in} F - H \subseteq G$. The collection of all soft ω -open sets in (X, τ, A) is denoted by τ_ω . It was proved in [2] that (X, τ_ω, A) is an STS with $\tau \subseteq \tau_\omega$.

In this study, using soft ω -open sets, we introduce a new concept in STSs, namely, ω^* -paracompactness, which extends ω^* -paracompactness in ordinary topological spaces. As symmetries, we provide characterizations of this concept. We also study the connection of soft ω^* -paracompactness with other related topological concepts. In particular, we show that soft ω^* -paracompactness and soft paracompactness are independent of each other. Moreover, we study the soft ω^* -paracompactness of the STS generated by an indexed family of ω^* -paracompact TSs.

2. Preliminaries

Herein, we recall several related definitions and results.

Definition 2.1. Let $(X, \mathfrak{S})$ be a TS.

(a) The set of all covers of $(X, \mathfrak{S})$ is denoted by $C(X, \mathfrak{S})$ and defined by

$$C(X, \mathfrak{S}) = \{\mathcal{B} \subseteq \mathcal{P}(X) : \cup \{B : B \in \mathcal{B}\} = X\}.$$

(b) The set of all open covers of $(X, \mathfrak{S})$ is denoted by $OC(X, \mathfrak{S})$ and defined by

$$OC(X, \mathfrak{S}) = \{\mathcal{B} \subseteq \mathfrak{S} : \cup \{B : B \in \mathcal{B}\} = X\}.$$

We recall that if $\mathcal{A}, \mathcal{B} \in C(X, \mathfrak{S})$, where $(X, \mathfrak{S})$ is a TS, then $\mathcal{A}$ is called a refinement of $\mathcal{B}$ (denoted as $\mathcal{A} \preceq \mathcal{B}$) if for each $A \in \mathcal{A}$, there is $B \in \mathcal{B}$ such that $A \subseteq B$. Moreover, let $(X, \mathfrak{S})$ be a TS and $\mathcal{A} \subseteq \mathcal{P}(X)$; then, $\mathcal{A}$ is called locally finite in $(X, \mathfrak{S})$ if for each $x \in X$, there is $U \in \mathfrak{S}$ such that $x \in U$ and $\{A \in \mathcal{A} : U \cap A \neq \emptyset\}$ is finite.

Definition 2.2 ([36]). A TS $(X, \mathfrak{S})$ is called ω^* -paracompact if for every $\mathcal{B} \in OC(X, \mathfrak{S}_\omega)$, there is $\mathcal{A} \in OC(X, \mathfrak{S}_\omega)$ such that $\mathcal{A}$ is locally finite in $(X, \mathfrak{S})$, and $\mathcal{A}$ is a refinement of $\mathcal{B}$.

Definition 2.3. Let (X, τ, A) be an STS.

(a) The set of all soft covers of (X, τ, A) is denoted by $SC(X, \tau, A)$ and defined by

$$SC(X, \tau, A) = \{\mathcal{H} \subseteq SS(X, A) : \tilde{\cup} \{H : H \in \mathcal{H}\} = 1_A\}.$$

(b) The set of all soft open covers of (X, τ, A) is denoted by $SOC(X, \tau, A)$ and defined by

$$SOC(X, \tau, A) = \{\mathcal{H} \subseteq \tau : \tilde{\cup} \{H : H \in \mathcal{H}\} = 1_A\}.$$

(c) The set of all soft closed covers of (X, τ, A) is denoted by $SCC(X, \mathfrak{S})$ and defined by

$$SCC(X, \tau, A) = \{\mathcal{H} : \mathcal{H} \subseteq \tau^c : \tilde{\cup} \{H : H \in \mathcal{H}\} = 1_A\}.$$

Definition 2.4 ([37]). Let (X, τ, A) be an STS and $\mathcal{H}, \mathcal{M} \subseteq SS(X, A)$.

(a) $\mathcal{H}$ is called soft locally finite in (X, τ, A) if for each $a_x \in SP(X, A)$, there is $F \in \tau$ such that $a_x \tilde{\in} F$, and $\{H \in \mathcal{H} : F \tilde{\cap} H \neq 0_A\}$ is finite.

(b) $\mathcal{M}$ is called a soft refinement of $\mathcal{H}$ (denoted as $\mathcal{M} \preceq \mathcal{H}$) if for each $M \in \mathcal{M}$, there is $H \in \mathcal{H}$ such that $M \subseteq H$.

Definition 2.5 ([37]). An STS (X, τ, A) is called soft paracompact if for every $\mathcal{H} \in SOC(X, \tau, A)$, there is $\mathcal{M} \in SOC(X, \tau, A)$ such that $\mathcal{M}$ is soft locally finite in (X, τ, A) and $\mathcal{M} \preceq \mathcal{H}$.

Definition 2.6 ([4]). An STS (X, τ, A) is called soft regular if for every $F \in \tau^c$ and $a_x \notin F$, there exist $U, V \in \tau$ such that $a_x \tilde{\in} U$, $F \subseteq V$, and $U \tilde{\cap} V = 0_A$.

Proposition 2.7 ([4]). An STS (X, τ, A) is soft regular if for every $V \in \tau$ and $a_x \tilde{\in} U$, there exists $V \in \tau$ such that $a_x \tilde{\in} V \subseteq Cl_\tau(V) \subseteq U$.

Proposition 2.8 ([37]). Let (X, τ, A) be an STS, where $A = \{a\}$. Then, (X, τ, A) is soft paracompact if and only if the TS (X, τ_a) is paracompact.

Proposition 2.9 ([2]). Let (X, τ, A) be an STS. Then, for all $a \in A$, we have $(\tau_a)_\omega = (\tau_\omega)_a$.

Proposition 2.10 ([2]). Let X be an initial universe, let A be a set of parameters, and let $\{\mathfrak{S}_a : a \in A\}$ be an indexed family of topologies on X . Then, $\left(\bigoplus_{a \in A} \mathfrak{S}_a\right)_\omega = \bigoplus_{a \in A} (\mathfrak{S}_a)_\omega$.

3. Soft ω^* -Paracompactness

This study is primarily concerned with the following concept.

Definition 3.1. An STS (X, τ, A) is called soft ω^* -paracompact if for every $\mathcal{H} \in SOC(X, \tau_\omega, A)$, there is $\mathcal{M} \in SOC(X, \tau_\omega, A)$ such that $\mathcal{M}$ is soft locally finite in (X, τ, A) and $\mathcal{M} \preceq \mathcal{H}$.

The following three Lemmas are used in Theorem 3.6 below.

Lemma 3.2. If $\mathcal{H}$ is soft locally finite in (X, τ, A) , then $\{Cl_{\tau_\omega}(H) : H \in \mathcal{H}\}$ is soft locally finite in (X, τ, A) .

Proof. Let $a_x \in SP(X, A)$; then, there is $F \in \tau$ such that $a_x \in F$, and $\{H \in \mathcal{H} : F \tilde{\cap} H \neq 0_A\}$ is finite.

Claim. $\{H \in \mathcal{H} : F \tilde{\cap} Cl_{\tau_\omega}(H) \neq 0_A\} \subseteq \{H \in \mathcal{H} : F \tilde{\cap} H \neq 0_A\}$.

Proof of Claim. We assume that $F \tilde{\cap} Cl_{\tau_\omega}(H) \neq 0_A$ for some $H \in \mathcal{H}$. We select $a_x \tilde{\in} F \tilde{\cap} Cl_{\tau_\omega}(H) \subseteq F \tilde{\cap} Cl_\tau(H)$; then, $a_x \tilde{\in} F \tilde{\cap} Cl_\tau(H)$, and thus $F \tilde{\cap} H \neq 0_A$. $\square$

Lemma 3.3. If $\mathcal{H}$ is soft locally finite in (X, τ, A) , then $\mathcal{H}$ is soft locally finite in (X, τ_ω, A) .

Proof. Let $a_x \in SP(X, A)$; then, there is $F \in \tau \subseteq \tau_\omega$ such that $a_x \in F$, and $\{H \in \mathcal{H} : F \tilde{\cap} H \neq 0_A\}$ is finite. $\square$

The following example shows that the converse of the implication in Lemma 3.3 is not true in general.

Example 3.4. Let $X = \mathbb{N}$, $\tau = \{0_A, 1_A\}$, $A = \mathbb{R}$, and $\mathcal{H} = SP(X, A)$. Then, $\mathcal{H}$ is soft locally finite in (X, τ_ω, A) , but it is not soft locally finite in (X, τ, A) .

Lemma 3.5. If $\mathcal{H}$ is soft locally finite in (X, τ, A) , then

$$\tilde{\cup} \{Cl_{\tau_\omega}(H) : H \in \mathcal{H}\} = Cl_{\tau_\omega}(\tilde{\cup} \{H : H \in \mathcal{H}\}).$$

Proof. We assume that $\mathcal{H}$ is soft locally finite in (X, τ, A) . By Lemma 3.3, $\mathcal{H}$ is soft locally finite in (X, τ_ω, A) . Thus, by applying Proposition 5.2 in [37], we obtain the result. $\square$

The following provides two characterizations of soft ω^* -paracompact STSs.

Theorem 3.6. Let (X, τ, A) be an STS so that (X, τ_ω, A) is soft regular. Then, the following are equivalent:

- (a) (X, τ, A) is soft ω^* -paracompact.
- (b) For every $\mathcal{H} \in SOC(X, \tau_\omega, A)$, there is $\mathcal{M} \in SC(X, \tau_\omega, A)$ such that $\mathcal{M}$ is soft locally finite in (X, τ, A) and $\mathcal{M} \tilde{\subseteq} \mathcal{H}$.
- (c) For every $\mathcal{H} \in SOC(X, \tau_\omega, A)$, there is $\mathcal{M} \in SCC(X, \tau_\omega, A)$ such that $\mathcal{M}$ is soft locally finite in (X, τ, A) and $\mathcal{M} \tilde{\subseteq} \mathcal{H}$.

Proof. (a) $\implies$ (b): Obvious.

(b) $\implies$ (c): We assume that (X, τ, A) is soft ω^* -paracompact, and let $\mathcal{H} \in SOC(X, \tau_\omega, A)$. For every $a_x \in SP(X, A)$, we select $H_{a_x} \in \mathcal{H}$ such that $a_x \tilde{\in} H_{a_x}$. As (X, τ_ω, A) is soft regular, by Proposition 2.7, for every $a_x \in SP(X, A)$, there is $N_{a_x} \in \tau$ such that $a_x \tilde{\in} N_{a_x} \subseteq Cl_{\tau_\omega}(N_{a_x}) \subseteq H_{a_x}$. Let $\mathcal{N} = \{N_{a_x} : a_x \in SP(X, A)\}$; then, $\mathcal{N} \in SOC(X, \tau_\omega, A)$. As (X, τ, A) is soft ω^* -paracompact, there is $\mathcal{M} \in SOC(X, \tau_\omega, A)$ such that $\mathcal{M}$ is soft locally finite in (X, τ, A) and $\mathcal{M} \tilde{\subseteq} \mathcal{N}$. Let $\mathcal{M}_1 = \{Cl_{\tau_\omega}(M) : M \in \mathcal{M}\}$. Then, $\mathcal{M}_1 \in SCC(X, \tau_\omega, A)$, and by Lemma 3.2, $\mathcal{M}_1$ is soft locally finite.

Claim (i). $\mathcal{M}_1 \tilde{\subseteq} \mathcal{H}$.

Proof of Claim. Let $Cl_{\tau_\omega}(M) \in \mathcal{M}_1$, where $M \in \mathcal{M}$. As $\mathcal{M} \tilde{\subseteq} \mathcal{N}$, there is $a_x \in SP(X, A)$ such that $M \subseteq N_{a_x}$,

and thus $Cl_{\tau_\omega}(M) \subseteq Cl_{\tau_\omega}(N_{a_x}) \subseteq H_{a_x}$ with $H_{a_x} \in \mathcal{H}$. It follows that $\mathcal{M}_1 \tilde{\subseteq} \mathcal{H}$.

(c) $\implies$ (a): Let $\mathcal{H} \in SOC(X, \tau_\omega, A)$. By (c), there is $\mathcal{L} \in SCC(X, \tau_\omega, A)$ such that $\mathcal{L}$ is soft locally finite in (X, τ, A) and $\mathcal{L} \tilde{\subseteq} \mathcal{H}$. As $\mathcal{L}$ is soft locally finite, for each $a_x \in SP(X, A)$, there is $N_{a_x} \in \tau$ such that $a_x \tilde{\in} N_{a_x}$, and $\{L \in \mathcal{L} : N_{a_x} \tilde{\cap} L \neq 0_A\}$ is finite. Let $\mathcal{N} = \{N_{a_x} : a_x \in SP(X, A)\}$. Then, $\mathcal{N} \in SOC(X, \tau, A) \subseteq SOC(X, \tau_\omega, A)$, and by (c), there is $\mathcal{G} \in SCC(X, \tau_\omega, A)$ such that $\mathcal{G}$ is soft locally finite in (X, τ, A) and $\mathcal{G} \tilde{\subseteq} \mathcal{N}$. As $\mathcal{L} \tilde{\subseteq} \mathcal{H}$, for every $L \in \mathcal{L}$, there is $H_L \in \mathcal{H}$ such that $L \subseteq H_L$. For every $L \in \mathcal{L}$, let $K_L = 1_A - \tilde{\cup} \{G \in \mathcal{G} : G \tilde{\cap} L = 0_A\}$ and $M_L = K_L \tilde{\cap} H_L$.

Claim (ii). (1) $\{K_L : L \in \mathcal{L}\} \subseteq \tau_\omega$.

(2) For every $L \in \mathcal{L}$, we have $L \subseteq K_L$.

(3) For every $L \in \mathcal{L}$ and $G \in \mathcal{G}$, $K_L \tilde{\cap} G \neq 0_A$ if and only if $L \tilde{\cap} G \neq 0_A$.

(4) $\{K_L : L \in \mathcal{L}\}$ is soft locally finite.

(5) $\{M_L : L \in \mathcal{L}\} \in SOC(X, \tau_\omega, A)$.

(6) $\{M_L : L \in \mathcal{L}\} \tilde{\subseteq} \mathcal{H}$.

(7) $\{M_L : L \in \mathcal{L}\}$ is soft locally finite.

Proof of Claim. 1. Follows from Lemma 3.5.

2. We assume toward a contradiction that there are $L_0 \in \mathcal{L}$ and $a_x \in SP(X, \tau)$ such that $a_x \tilde{\in} L_0 \tilde{\cap} (\tilde{\cup} \{G \in \mathcal{G} : G \tilde{\cap} L_0 = 0_A\})$. Then, $a_x \tilde{\in} L_0$ and $a_x \tilde{\in} (\tilde{\cup} \{G \in \mathcal{G} : G \tilde{\cap} L_0 = 0_A\})$. As $a_x \tilde{\in} (\tilde{\cup} \{G \in \mathcal{G} : G \tilde{\cap} L_0 = 0_A\})$, there is $G_x \in \mathcal{G}$ such that $G_x \tilde{\cap} L_0 = 0_A$ and $a_x \tilde{\in} G_x$. However, $a_x \tilde{\in} G_x \tilde{\cap} L_0$, which is a contradiction.

3. *Necessity.* Let $L \in \mathcal{L}$ and $G \in \mathcal{G}$ such that $K_L \tilde{\cap} G \neq 0_A$, and we assume toward a contradiction that $L \tilde{\cap} G = 0_A$. We select $a_x \tilde{\in} K_L \tilde{\cap} G$. Then, $a_x \tilde{\in} K_L$ and $a_x \tilde{\in} G$. As $L \tilde{\cap} G = 0_A$ and $a_x \tilde{\in} G$, we have $a_x \tilde{\in} \tilde{\cup} \{G \in \mathcal{G} : G \tilde{\cap} L = 0_A\}$ and so $a_x \notin K_L$, which is a contradiction.

Sufficiency. Let $L \in \mathcal{L}$ and $G \in \mathcal{G}$ such that $L \tilde{\cap} G \neq 0_A$. By (2), $L \subseteq K_L$, and therefore $L \tilde{\cap} G \subseteq K_L \tilde{\cap} G$. It follows that $K_L \tilde{\cap} G \neq 0_A$.

4. Let $a_x \in SP(X, \tau)$. As $\mathcal{G}$ is soft locally finite, there is $S \in \tau$ such that $a_x \tilde{\in} S$, and $\{G \in \mathcal{G} : G \tilde{\cap} S \neq 0_A\}$ is finite; let $\{G \in \mathcal{G} : G \tilde{\cap} S \neq 0_A\} = \{G_1, G_2, \dots, G_n\}$. We assume that for some $L \in \mathcal{L}$, we have $K_L \tilde{\cap} S \neq 0_A$. We select $b_y \tilde{\in} K_L \tilde{\cap} S$. As $b_y \tilde{\in} 1_A = \tilde{\cup} \{G : G \in \mathcal{G}\}$, there is $G \in \mathcal{G}$ such that $b_y \tilde{\in} G$. As $b_y \tilde{\in} K_L$, by (3), we have $G \tilde{\cap} L \neq 0_A$. As $b_y \tilde{\in} G \tilde{\cap} S$, we have $G = G_i$ for some $i = 1, 2, \dots, n$. As $\mathcal{G} \tilde{\subseteq} \mathcal{N}$, for every $i = 1, 2, \dots, n$, there is $(a_i)_{x_i}$ such that $G_i \subseteq N_{(a_i)_{x_i}}$. Therefore, $\{L \in \mathcal{L} : K_L \tilde{\cap} S \neq 0_A\}$ is finite. It follows that $\{K_L : L \in \mathcal{L}\}$ is soft locally finite.

5. Let $L \in \mathcal{L}$. As $H_L \in \mathcal{H} \subseteq \tau_\omega$, and by (1), we have $K_L \in$

τ_ω , it follows that $M_L = K_L \tilde{\cap} H_L \in \tau_\omega$. However, by (2) L . Moreover, $L \subseteq H_L$, and by (2), we have $L \subseteq K_L$; accordingly, $L \subseteq M_L$. It follows that $\{M_L : L \in \mathcal{L}\} \subseteq \tau_\omega$ and $1_A = \bigcup \{L : L \in \mathcal{L}\} \subseteq \bigcup \{M_L : L \in \mathcal{L}\}$. Therefore, $\{M_L : L \in \mathcal{L}\} \in SOC(X, \tau_\omega, A)$.

6. Let $L \in \mathcal{L}$. Then, $M_L = K_L \tilde{\cap} H_L \subseteq H_L$ with $H_L \in \mathcal{H}$. Thus, $\{M_L : L \in \mathcal{L}\} \preceq \mathcal{H}$.

7. This follows directly from (4). $\square$

The following characterization of soft paracompact STSs is used in the proof of Theorem 3.8.

Proposition 3.7. Let (X, τ, A) be an STS, and let $\mathcal{B}$ be a soft base of τ . Then, (X, τ, A) is soft paracompact if and only if for every $\mathcal{H} \in SOC(X, \tau, A)$ with $\mathcal{H} \subseteq \mathcal{B}$, there is $\mathcal{M} \in SOC(X, \tau, A)$ such that $\mathcal{M}$ is soft locally finite in (X, τ, A) and $\mathcal{M} \preceq \mathcal{H}$.

Proof. Straightforward. $\square$

Theorem 3.8. Let X be an initial universe, and let A be a set of parameters. Moreover, let $\{\mathfrak{S}_a : a \in A\}$ be an indexed family of topologies on X . Then, $\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$ is soft paracompact if and only if $(X, \mathfrak{S}_a)$ is paracompact for all $a \in A$.

Proof. Necessity. We assume that $\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$ is soft paracompact. Let $b \in A$. To show that $(X, \mathfrak{S}_b)$ is paracompact, let $\mathcal{U} \subseteq OC(X, \mathfrak{S}_b)$. We set $\mathcal{H} = \{b_U : U \in \mathcal{U}\} \cup \{a_X : a \in A - \{b\}\}$. Then, $\mathcal{H} \in SOC\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$. As $\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$ is soft paracompact, there is $\mathcal{M} \in SOC\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$ such that $\mathcal{M}$ is soft locally finite in $\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$ and $\mathcal{M} \preceq \mathcal{H}$. Let $\mathcal{M}_b = \{M(b) : M \in \mathcal{M}\}$.

Claim. (1) $\mathcal{M}_b \in OC(X, \mathfrak{S}_b)$.

(2) $\mathcal{M}_b$ is locally finite in $(X, \mathfrak{S}_b)$.

(3) $\mathcal{M}_b \preceq \mathcal{U}$.

Proof of Claim. 1. As $\mathcal{M} \subseteq \bigoplus_{a \in A} \mathfrak{S}_a$, by the definition of $\mathfrak{S}_b$, we have $\mathcal{M}_b \subseteq \mathfrak{S}_b$. As $\mathcal{M} \in SOC\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$, we have $\bigcup \{M : M \in \mathcal{M}\} = 1_A$, and thus $\left(\bigcup \{M : M \in \mathcal{M}\}\right)(b) = \bigcup \{M(b) : M \in \mathcal{M}\} = X$. Therefore, $\mathcal{M}_b \in OC(X, \mathfrak{S}_b)$.

2. Let $x \in X$. Then, $b_x \in SP(X, A)$. As $\mathcal{M}$ is soft locally finite in $\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$, there is $F \in \bigoplus_{a \in A} \mathfrak{S}_a$ such that $b_x \tilde{\subseteq} F$, and $\{M \in \mathcal{M} : F \tilde{\cap} M \neq 0_A\}$ is finite. Thus, $x \in F(b)$

$\in \mathfrak{S}_b$. If $M \in \mathcal{M}$ is such that $F(b) \cap M(b) \neq \emptyset$, then $F \tilde{\cap} M \neq 0_A$. It follows that $\mathcal{M}_b$ is locally finite in $(X, \mathfrak{S}_b)$.

3. Let $M(b) \in \mathcal{M}_b - \{\emptyset\}$, where $M \in \mathcal{M}$. As $\mathcal{M} \preceq \mathcal{H}$, there is $H \in \mathcal{H}$ such that $M \subseteq H$, and thus $M(b) \subseteq H(b)$. As $M(b) \neq \emptyset$, we have $H(b) \neq \emptyset$, and thus there is $U \in \mathcal{U}$ such that $H = b_U$ and $H(b) = U$. It follows that $\mathcal{M}_b \preceq \mathcal{U}$.

Sufficiency. We assume that $(X, \mathfrak{S}_a)$ is paracompact for all $a \in A$. Let $\mathcal{B} = \{a_Y : a \in A \text{ and } Y \in \mathfrak{S}_a\}$. By Theorem 3.5 in [1], $\mathcal{B}$ is a soft base of $\bigoplus_{a \in A} \mathfrak{S}_a$. We apply Proposition

3.7. Let $\mathcal{H} \in SOC\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$ with $\mathcal{H} \subseteq \mathcal{B}$. For each $a \in A$, let $\mathcal{H}_a = \{Y \subseteq X : a_Y \in \mathcal{H}\}$. Then, for all $a \in A$, $\mathcal{H}_a \in OC(X, \mathfrak{S}_a)$, where $(X, \mathfrak{S}_a)$ is paracompact, and thus there is $\mathcal{M}_a \in OC(X, \mathfrak{S}_a)$ such that $\mathcal{M}_a$ is locally finite in $(X, \mathfrak{S}_a)$ and $\mathcal{M}_a \preceq \mathcal{H}_a$. Let $\mathcal{G} = \{a_Y : a \in A \text{ and } Y \in \mathcal{M}_a\}$.

Claim. (1) $\mathcal{G} \in SOC\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$.

(2) $\mathcal{G}$ is soft locally finite in $\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$.

(3) $\mathcal{G} \preceq \mathcal{H}$.

Proof of Claim. 1. For all $a \in A$, we have $\mathcal{M}_a \subseteq \mathfrak{S}_a$, and therefore $\mathcal{G} \subseteq \bigoplus_{a \in A} \mathfrak{S}_a$. For all $a \in A$, we have $\mathcal{M}_a \in OC(X, \mathfrak{S}_a)$, and thus $\left(\bigcup \{a_Y : a \in A \text{ and } Y \in \mathcal{M}_a\}\right)(a) = \bigcup \{Y : Y \in \mathcal{M}_a\} = X$. Therefore, $\bigcup \{a_Y : a \in A \text{ and } Y \in \mathcal{M}_a\} = 1_A$. It follows that $\mathcal{G} \in SOC\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$.

2. Let $b_x \in SP(X, A)$. As $\mathcal{M}_a$ is locally finite in $(X, \mathfrak{S}_b)$, there is $O \in \mathfrak{S}_b$ such that $x \in O$, and $\{Y : O \cap Y \neq \emptyset\}$ is finite.

We have $b_x \in b_O \in \bigoplus_{a \in A} \mathfrak{S}_a$. If $b_O \tilde{\cap} a_Y \neq 0_A$, then $a = b$ and $O \cap Y \neq \emptyset$. This shows that $\{a_Y : a \in A \text{ and } Y \in \mathcal{M}_a, \text{ and } b_O \tilde{\cap} a_Y \neq 0_A\}$ is finite. It follows that $\mathcal{G}$ is soft locally finite in $\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$.

3. Let $a_Y \in \mathcal{G}$ with $a \in A$ and $Y \in \mathcal{M}_a$. As $\mathcal{M}_a \preceq \mathcal{H}_a$, there is $Z \in \mathcal{H}_a$, where $a_Z \in \mathcal{H}$, such that $Y \subseteq Z$. Therefore, $a_Y \subseteq a_Z$. It follows that $\mathcal{G} \preceq \mathcal{H}$. $\square$

Proposition 3.9. Let (X, τ, A) be an STS, and let $\mathcal{B}$ be a soft base of τ_ω . Then, (X, τ, A) is soft ω^* -paracompact if and only if for every $\mathcal{H} \in SOC(X, \tau_\omega, A)$ with $\mathcal{H} \subseteq \mathcal{B}$, there is $\mathcal{M} \in SOC(X, \tau_\omega, A)$ such that $\mathcal{M}$ is soft locally finite in (X, τ, A) and $\mathcal{M} \preceq \mathcal{H}$.

Proof. Straightforward. $\square$

Corollary 3.10. Let (X, τ, A) be an STS. Then, (X, τ, A) is soft ω^* -paracompact if and only if for every $\mathcal{H} \in SOC(X, \tau_\omega, A)$

with $\mathcal{H} \subseteq \{F - H : F \in \tau \text{ and } H \in CSS(X, A)\}$, there is $\mathcal{M} \in SOC(X, \tau_\omega, A)$ such that $\mathcal{M}$ is soft locally finite in (X, τ, A) and $\mathcal{M} \tilde{\preceq} \mathcal{H}$.

Lemma 3.11. Let (X, τ, A) be an STS. If $\mathcal{B}$ is a soft base for τ , then $\{B - H : B \in \mathcal{B} \text{ and } H \in CSS(X, A)\}$ is a soft base for τ_ω .

Proof. Straightforward. $\square$

Theorem 3.12. Let (X, τ, A) be an STS, where $A = \{a\}$. Then, (X, τ, A) is soft ω^* -paracompact if and only if the TS (X, τ_a) is ω^* -paracompact.

Proof. Necessity. We assume that (X, τ, A) is soft ω^* -paracompact. Let $\mathcal{U} \in OC(X, (\tau_a)_\omega)$. By Proposition 2.9, we have $(\tau_a)_\omega = (\tau_\omega)_a$, and thus $\mathcal{U} \in OC(X, (\tau_\omega)_a)$.

Then, $\{a_U : U \in \mathcal{U}\} \in SOC(X, \tau_\omega, A)$. As (X, τ, A) is soft ω^* -paracompact, there is $\mathcal{H} \in SOC(X, \tau_\omega, A)$ such that $\mathcal{H}$ is soft locally finite in (X, τ, A) and $\mathcal{H} \tilde{\preceq} \{a_U : U \in \mathcal{U}\}$. Let $\mathcal{V} = \{H(a) : H \in \mathcal{H}\}$.

Claim. (1) $\mathcal{V} \in OC(X, (\tau_a)_\omega)$.

(2) $\mathcal{V}$ is locally finite in (X, τ_a) .

(3) $\mathcal{V} \preceq \mathcal{U}$.

Proof of Claim. 1. As $\mathcal{H} \subseteq \tau_\omega$, we have $\mathcal{V} \subseteq (\tau_\omega)_a$ and by Proposition 2.9, it follows that $\mathcal{V} \subseteq (\tau_a)_\omega$. In addition, as $\mathcal{H} \in SOC(X, \tau_\omega, A)$, we have $\tilde{\cup}\{H : H \in \mathcal{H}\} = 1_A$, and thus $(\tilde{\cup}\{H : H \in \mathcal{H}\})(a) = \cup\{H(a) : H \in \mathcal{H}\} = X$. It follows that $\mathcal{V} \in OC(X, (\tau_a)_\omega)$.

2. Let $x \in X$. Then, $a_x \in SP(X, A)$. As $\mathcal{H}$ is soft locally finite in (X, τ, A) , there is $F \in \tau$ such that $a_x \tilde{\in} F$, and $\{H \in \mathcal{H} : F \tilde{\cap} H \neq 0_A\}$ is finite. Then, $x \in F(a) \in \tau_a$. Furthermore, if for $H \in \mathcal{H}$, we have $F(a) \cap H(a) \neq \emptyset$, then $F \tilde{\cap} H \neq 0_A$. It follows that $\{H \in \mathcal{H} : F(a) \cap H(a) \neq \emptyset\}$ is finite. Therefore, $\mathcal{V}$ is locally finite in (X, τ_a) .

3. Let $H(a) \in \mathcal{V}$, where $H \in \mathcal{H}$. As $\mathcal{H} \tilde{\preceq} \{a_U : U \in \mathcal{U}\}$, there is $U \in \mathcal{U}$ such that $H \tilde{\subseteq} a_U$; thus, $H(a) \subseteq U$. This shows that $\mathcal{V} \preceq \mathcal{U}$.

Sufficiency. We assume that (X, τ_a) is ω^* -paracompact. Let $\mathcal{H} \in SOC(X, \tau_\omega, A)$. Then, $\{H(a) : H \in \mathcal{H}\} \in OC(X, (\tau_\omega)_a) = OC(X, (\tau_a)_\omega)$. As (X, τ_a) is ω^* -paracompact, there is $\mathcal{V} \in OC(X, (\tau_a)_\omega) = OC(X, (\tau_\omega)_a)$ such that $\mathcal{V}$ is locally finite in (X, τ_a) and $\mathcal{V} \preceq \{H(a) : H \in \mathcal{H}\}$. Let $\mathcal{M} = \{a_V : V \in \mathcal{V}\}$.

Claim. (1) $\mathcal{M} \in SOC(X, \tau_\omega, A)$.

(2) $\mathcal{M}$ is soft locally finite in (X, τ, A) .

(3) $\mathcal{M} \tilde{\preceq} \mathcal{H}$.

Proof of Claim. 1. As $\mathcal{V} \subseteq (\tau_\omega)_a$, we have $\mathcal{M} \subseteq \tau_\omega$. In addition, as $\mathcal{V} \in OC(X, (\tau_\omega)_a)$, we have $\cup\{V : V \in \mathcal{V}\} = X$, and thus $\tilde{\cup}\{a_V : V \in \mathcal{V}\} = 1_A$. It follows that $\mathcal{M} \in SOC(X, \tau_\omega, A)$.

2. Let $a_x \in SP(X, A)$. As $\mathcal{V}$ is locally finite in (X, τ_a) , there is $O \in \tau_a$ such that $x \in O$ and $\{V \in \mathcal{V} : O \cap V \neq \emptyset\}$ is finite. Then, $a_x \tilde{\in} a_O \in \tau$. If $a_V \in \mathcal{M}$, where $V \in \mathcal{V}$ and $a_O \tilde{\cap} a_V \neq 0_A$, then $(a_O \tilde{\cap} a_V)(a) = O \cap V \neq \emptyset$. It follows that $\mathcal{M}$ is soft locally finite in $SOC(X, \tau, A)$.

3. Let $a_V \in \mathcal{M}$, where $V \in \mathcal{V}$. As $\mathcal{V} \preceq \{H(a) : H \in \mathcal{H}\}$, there is $H \in \mathcal{H}$ such that $V \subseteq H(a)$, and thus $a_V \tilde{\subseteq} H$. This shows that $\mathcal{M} \tilde{\preceq} \mathcal{H}$. $\square$

Theorem 3.13. Let X be an initial universe, and let A be a set of parameters. Moreover, let $\{\mathfrak{S}_a : a \in A\}$ be an indexed family of topologies on X . Then, $\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$ is soft ω^* -paracompact if and only if $(X, \mathfrak{S}_a)$ is ω^* -paracompact for all $a \in A$.

Proof. Necessity. We assume that $\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$ is soft ω^* -paracompact. Let $b \in A$. To show that $(X, \mathfrak{S}_b)$ is ω^* -paracompact, let $\mathcal{U} \subseteq OC(X, (\mathfrak{S}_b)_\omega)$. We set $\mathcal{H} = \{b_U : U \in \mathcal{U}\} \cup \{a_X : a \in A - \{b\}\}$. Then, $\mathcal{H} \in SOC\left(X, \bigoplus_{a \in A} (\mathfrak{S}_a)_\omega, A\right)$.

By Proposition 2.10, $\bigoplus_{a \in A} (\mathfrak{S}_a)_\omega = \left(\bigoplus_{a \in A} \mathfrak{S}_a\right)_\omega$, and thus $\mathcal{H} \in SOC\left(X, \left(\bigoplus_{a \in A} \mathfrak{S}_a\right)_\omega, A\right)$. As $\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$ is soft ω^* -paracompact, there is $\mathcal{M} \in SOC\left(X, \left(\bigoplus_{a \in A} \mathfrak{S}_a\right)_\omega, A\right)$ such

that $\mathcal{M}$ is soft locally finite in $\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$ and $\mathcal{M} \tilde{\preceq} \mathcal{H}$.

By Proposition 2.10, $\mathcal{M} \in SOC\left(X, \bigoplus_{a \in A} (\mathfrak{S}_a)_\omega, A\right)$. Let $\mathcal{M}_b = \{M(b) : M \in \mathcal{M}\}$.

Claim. (1) $\mathcal{M}_b \in OC(X, (\mathfrak{S}_b)_\omega)$.

(2) $\mathcal{M}_b$ is locally finite in $(X, \mathfrak{S}_b)$.

(3) $\mathcal{M}_b \preceq \mathcal{U}$.

Proof of Claim. 1. As $\mathcal{M} \subseteq \bigoplus_{a \in A} (\mathfrak{S}_a)_\omega$, by the definition of $\mathfrak{S}_b$, we have $\mathcal{M}_b \subseteq (\mathfrak{S}_b)_\omega$. As $\mathcal{M} \in SOC\left(X, \bigoplus_{a \in A} (\mathfrak{S}_a)_\omega, A\right)$, we have $\tilde{\cup}\{M : M \in \mathcal{M}\} = 1_A$; hence, $(\tilde{\cup}\{M : M \in \mathcal{M}\})(b) = \cup\{M(b) : M \in \mathcal{M}\} = X$. Therefore, $\mathcal{M}_b \in OC(X, (\mathfrak{S}_b)_\omega)$.

2. Let $x \in X$. Then, $b_x \in SP(X, A)$. As $\mathcal{M}$ is soft locally finite in $\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$, there is $F \in \bigoplus_{a \in A} \mathfrak{S}_a$ such that $b_x \tilde{\in} F$, and $\{M \in \mathcal{M} : F \tilde{\cap} M \neq 0_A\}$ is finite. Thus, $x \in F(b)$

$\in \mathfrak{S}_b$. If $M \in \mathcal{M}$ is such that $F(b) \cap M(b) \neq \emptyset$, then $F \tilde{\cap} M \neq 0_A$. It follows that $\mathcal{M}_b$ is locally finite in $(X, \mathfrak{S}_b)$.

3. Let $M(b) \in \mathcal{M}_b - \{\emptyset\}$, where $M \in \mathcal{M}$. As $\mathcal{M} \tilde{\supseteq} \mathcal{H}$, there is $H \in \mathcal{H}$ such that $M \subseteq H$, and thus $M(b) \subseteq H(b)$. As $M(b) \neq \emptyset$, we have $H(b) \neq \emptyset$, and thus there is $U \in \mathcal{U}$ such that $H = b_U$ and $H(b) = U$. It follows that $\mathcal{M}_b \preceq \mathcal{U}$.

Sufficiency. We assume that $(X, \mathfrak{S}_a)$ is ω^* -paracompact for all $a \in A$. Let $\mathcal{B} = \{a_Y : a \in A \text{ and } Y \in (\mathfrak{S}_a)_\omega\}$. Then, $\mathcal{B}$ is a soft base of $\bigoplus_{a \in A} (\mathfrak{S}_a)_\omega = \left(\bigoplus_{a \in A} \mathfrak{S}_a \right)_\omega$. We apply Proposition

3.9. Let $\mathcal{H} \in SOC \left(X, \left(\bigoplus_{a \in A} \mathfrak{S}_a \right)_\omega, A \right)$ with $\mathcal{H} \subseteq \mathcal{B}$. For each $a \in A$, let $\mathcal{H}_a = \{Y \subseteq X : a_Y \in \mathcal{H}\}$. Then, for all $a \in A$, we have $\mathcal{H}_a \in OC \left(X, (\mathfrak{S}_a)_\omega \right)$, where $(X, \mathfrak{S}_a)$ is ω^* -paracompact. Thus, there is $\mathcal{M}_a \in OC \left(X, (\mathfrak{S}_a)_\omega \right)$ such that $\mathcal{M}_a$ is locally finite in $(X, \mathfrak{S}_a)$ and $\mathcal{M}_a \preceq \mathcal{H}_a$. Let $\mathcal{G} = \{a_Y : a \in A \text{ and } Y \in \mathcal{M}_a\}$.

Claim. (1) $\mathcal{G} \in SOC \left(X, \left(\bigoplus_{a \in A} \mathfrak{S}_a \right)_\omega, A \right)$.

(2) $\mathcal{G}$ is soft locally finite in $\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A \right)$.

(3) $\mathcal{G} \tilde{\supseteq} \mathcal{H}$.

Proof of Claim. 1. For all $a \in A$, we have $\mathcal{M}_a \subseteq (\mathfrak{S}_a)_\omega$, and therefore $\mathcal{G} \subseteq \bigoplus_{a \in A} (\mathfrak{S}_a)_\omega = \left(\bigoplus_{a \in A} \mathfrak{S}_a \right)_\omega$. For all $a \in A$, we have $\mathcal{M}_a \in OC \left(X, (\mathfrak{S}_a)_\omega \right)$, and thus $\bigcup \{a_Y : a \in A \text{ and } Y \in \mathcal{M}_a\} (a) = \bigcup \{Y : Y \in \mathcal{M}_a\} = X$. Therefore, $\bigcup \{a_Y : a \in A \text{ and } Y \in \mathcal{M}_a\} = 1_A$. It follows that $\mathcal{G} \in SOC \left(X, \left(\bigoplus_{a \in A} \mathfrak{S}_a \right)_\omega, A \right)$.

2. Let $b_x \in SP(X, A)$. As $\mathcal{M}_a$ is locally finite in $(X, \mathfrak{S}_b)$, there is $O \in \mathfrak{S}_b$ such that $x \in O$, and $\{Y : O \cap Y \neq \emptyset\}$ is finite. We have $b_x \in b_O \in \bigoplus_{a \in A} \mathfrak{S}_a$. If $b_O \tilde{\cap} a_Y \neq 0_A$, then $a = b$ and $O \cap Y \neq \emptyset$. This shows that $\{a_Y : a \in A \text{ and } Y \in \mathcal{M}_a, \text{ and } b_O \tilde{\cap} a_Y \neq 0_A\}$ is finite. It follows that $\mathcal{G}$ is soft locally finite in $\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A \right)$.

3. Let $a_Y \in \mathcal{G}$ with $a \in A$ and $Y \in \mathcal{M}_a$. As $\mathcal{M}_a \preceq \mathcal{H}_a$, there is $Z \in \mathcal{H}_a$, where $a_Z \in \mathcal{H}$, such that $Y \subseteq Z$. Therefore, $a_Y \subseteq a_Z$. It follows that $\mathcal{G} \tilde{\supseteq} \mathcal{H}$. $\square$

4. Soft ω^* -Paracompactness versus Soft Paracompactness

Herein, we show by examples that soft ω^* -paracompactness and soft paracompactness are independent of each other.

Theorem 4.1. Every soft regular soft ω^* -paracompact STS is soft paracompact.

Proof. Let (X, τ, A) be soft regular and soft ω^* -paracompact. Let $\mathcal{H} \in SOC(X, \tau, A) \subseteq SOC(X, \tau_\omega, A)$. Then, $\mathcal{H} \in$

$SOC(X, \tau_\omega, A)$. As (X, τ, A) is soft ω^* -paracompact, there is $\mathcal{M} \in SOC(X, \tau_\omega, A)$ such that $\mathcal{M}$ is soft locally finite in (X, τ, A) and $\mathcal{M} \tilde{\supseteq} \mathcal{H}$. It is clear that $\mathcal{M} \in SC(X, \tau, A)$. As (X, τ, A) is soft regular, by Theorem 5.14 in [37], it follows that (X, τ, A) is soft paracompact. $\square$

Theorem 4.2. If (X, τ, A) is a soft ω^* -paracompact STS, then (X, τ_ω, A) is soft paracompact.

Proof. We assume that (X, τ, A) is soft ω^* -paracompact. Let $\mathcal{H} \in SOC(X, \tau_\omega, A)$. As (X, τ, A) is soft ω^* -paracompact, there is $\mathcal{M} \in SOC(X, \tau_\omega, A)$ such that $\mathcal{M}$ is soft locally finite in (X, τ, A) and $\mathcal{M} \tilde{\supseteq} \mathcal{H}$. As $\mathcal{M}$ is soft locally finite in (X, τ, A) , we have that $\mathcal{M}$ is soft locally finite in (X, τ_ω, A) . It follows that (X, τ_ω, A) is soft paracompact. $\square$

Definition 4.3. An STS (X, τ, A) is called strongly soft locally finite if for every $b_x \in SP(X, A)$, there exists $H \in \tau$ such that $Supp(H)$ is finite, $b_x \tilde{\subseteq} H$, and $H(a)$ is finite for all $a \in A$.

Lemma 4.4. Let (X, τ, A) be a strongly soft locally finite STS, and let $\mathcal{H} \subseteq SP(X, A)$. Then, $\mathcal{H}$ is soft locally finite in (X, τ, A) .

Proof. We assume that (X, τ, A) is a strongly soft locally finite, and let $\mathcal{H} \subseteq SP(X, A)$. Let $b_x \in SP(X, A)$. As (X, τ, A) is strongly soft locally finite, there exists $G \in \tau$ such that $Supp(G)$ is finite, $b_x \tilde{\subseteq} G$, and $G(a)$ is finite for all $a \in A$. It is not difficult to verify that $\{a_x : a_x \tilde{\subseteq} G\}$ is finite. Thus, $\{H \in \mathcal{H} : G \tilde{\cap} H \neq 0_A\}$ is finite. It follows that $\mathcal{H}$ is soft locally finite in (X, τ, A) . $\square$

Theorem 4.5. Every strongly soft locally finite STS is soft ω^* -paracompact.

Proof. We assume that (X, τ, A) is strongly soft locally finite. Let $\mathcal{H} \in SOC(X, \tau_\omega, A)$. Let $\mathcal{M} = SP(X, A)$. As (X, τ, A) is strongly soft locally finite, (X, τ, A) is soft locally countable, and by Corollary 14 in [2], we have that (X, τ_ω, A) is a discrete STS. Thus, $\mathcal{M} \in SOC(X, \tau_\omega, A)$. By Lemma 4.4, $\mathcal{M}$ is soft locally finite in (X, τ, A) . Moreover, it is clear that $\mathcal{M} \tilde{\supseteq} \mathcal{H}$. It follows that (X, τ_ω, A) is soft ω^* -paracompact. $\square$

Theorem 4.6. Every strongly soft locally countable soft ω^* -paracompact STS is strongly soft locally finite.

Proof. Let (X, τ, A) be strongly soft locally countable and soft ω^* -paracompact. Let $\mathcal{H} = SP(X, A)$. Then, by Corollary 14 in [2], (X, τ_ω, A) is a discrete STS, and thus $\mathcal{H} \in SOC(X, \tau_\omega, A)$. As (X, τ, A) is soft ω^* -paracompact, there is

$\mathcal{M} \in SOC(X, \tau_\omega, A)$ such that $\mathcal{M}$ is soft locally finite in (X, τ, A) , $\mathcal{M} \tilde{\subseteq} \mathcal{H}$, and for every $M \in \mathcal{M}$, we have $M \neq 0_A$. For every $M \in \mathcal{M}$, there is a_{x_M} such that $M \tilde{\subseteq} a_{x_M}$, and thus $M = a_{x_M}$. Therefore, $\mathcal{M} \subseteq SP(X, A)$. As $\mathcal{M} \in SOC(X, \tau_\omega, A)$, we have $\mathcal{M} = SP(X, A)$. As $\mathcal{M}$ is soft locally finite in (X, τ, A) , for every $a_x \in SP(X, A)$, there is $G \in \tau$ such that $a_x \tilde{\subseteq} G$, and $\{M \in \mathcal{M} : G \tilde{\cap} M \neq 0_A\} = \{b_y \in SP(X, A) : G \tilde{\cap} b_y \neq 0_A\}$ is finite; thus, $Supp(G)$ is finite, and for every $b \in A$, $G(b)$ is finite. It follows that (X, τ, A) is strongly soft locally finite. $\square$

Theorem 4.7. If (X, τ, A) is soft separable and soft anti-locally countable, then (X, τ_ω, A) is not soft regular.

Proof. We assume toward a contradiction that (X, τ_ω, A) is soft regular. As (X, τ, A) is soft separable, there is $G \in SCSS(X, A)$ such that $1_A = Cl_\tau(G)$. As (X, τ, A) is strongly soft anti-locally countable, we have $1_A \notin SCSS(X, A)$; thus, $a_x \tilde{\subseteq} 1_A - G$. As (X, τ_ω, A) is soft regular and G is soft closed in (X, τ_ω, A) , there exist $U \in \tau_\omega, V \in \tau$ and $H \in CSS(X, A)$ such that $G \tilde{\subseteq} U$, $a_x \tilde{\subseteq} V - H$, and $U \tilde{\cap} (V - H) = 0_A$. As $1_A = Cl_\tau(G)$ and $a_x \tilde{\subseteq} V \in \tau$, there is $b_y \tilde{\subseteq} G \tilde{\cap} V$. As $b_y \tilde{\subseteq} G \tilde{\subseteq} U$, there exist $W \in \tau$ and $M \in CSS(X, A)$ such that $b_y \tilde{\subseteq} W - M \tilde{\subseteq} U$. As $b_y \tilde{\subseteq} U \tilde{\cap} W \in \tau$, we have $U \tilde{\cap} W \in \tau - \{0_A\}$. As (X, τ, A) is soft anti-locally countable, we have $U \tilde{\cap} W \notin CSS(X, A)$. As $U \tilde{\cap} W - H \tilde{\cap} G \tilde{\subseteq} U \tilde{\cap} (V - H) = 0_A$, we have $U \tilde{\cap} W \tilde{\subseteq} H \tilde{\cap} G$, and thus $U \tilde{\cap} W \in CSS(X, A)$, which is a contradiction. $\square$

Theorem 4.8. Every soft anti-locally countable soft T_2 soft paracompact soft separable STS is not soft ω^* -paracompact.

Proof. We assume that (X, τ, A) is soft anti-locally countable, soft T_2 , soft ω^* -paracompact, and soft separable. We assume toward a contradiction that (X, τ, A) is soft ω^* -paracompact. As (X, τ, A) is soft ω^* -paracompact, by Theorem 4.2 (X, τ_ω, A) is soft paracompact. As (X, τ, A) is soft T_2 , it is clear that (X, τ_ω, A) is soft T_2 . As (X, τ_ω, A) is soft T_2 and soft paracompact, by Theorem 5.8 in [37], (X, τ_ω, A) is soft regular. However, as (X, τ, A) is soft anti-locally countable and soft separable, by Theorem 4.1, (X, τ_ω, A) is not soft regular, which is a contradiction. $\square$

Example 4.9. Let $A = \{a\}$, $X = \mathbb{R}$, and $\tau = \{F \in SS(X, A) : F(a) \text{ belongs to the usual topology on } \mathbb{R}\}$. Then, clearly, (X, τ, A) is soft anti-locally countable, soft T_2 , soft paracompact, and soft separable. Thus, by Theorem 4.8, (X, τ, A) is not soft ω^* -paracompact.

Example 4.10. Let $(X, \mathfrak{S})$ be the irrational slope topology (Example 75 in [38]). Let $A\{a\}$ and $\tau = \{F \in SS(X, A) : F(a) \in \mathfrak{S}\}$. Clearly, (X, τ_ω, A) is a discrete STS. However, by Theorem 4.6, (X, τ_ω, A) is not soft ω^* -paracompact.

5. Conclusion

We defined and studied a new concept STSs, soft ω^* -paracompactness. Several characterizations and relationships regarding this new concept were proved. In the future, we intend to provide more characterizations and prove soft mapping theorems for soft ω^* -paracompactness, as well as define new soft topological concepts using soft ω -open sets.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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