On Soft Topological Polygroups and Their Examples

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Abstract

Polygroups are a special subclass of hypergroups that satisfies group-like axioms. A polygroup is a multigroup that is completely regular and reversible in itself. A topological polygroup is a polygroup P with a topology on P that satisfies certain conditions. Moreover, the concept of soft sets is a general mathematical tool for dealing with uncertainty. In this study, we investigate soft topological polygroups over a polygroup. The ideas presented in this article can be used to build more polygroups and soft topological polygroups.

Keywords: Soft set, Topological polygroup, Soft polygroup, Soft topological polygroups

1. Introduction

The real world is full of uncertainties that are not supported by classical mathematical structures. Therefore, it is necessary to redefine these structures to include uncertainties. Theory of fuzzy sets [1], vague set theory, rough set theory, interval mathematics theory, and other mathematical tools can help us implement these ideas. Much effort has been made on this subject in previous research. For example, Molodstov [2] introduced the concept of soft sets, Feng et al. [3–5] extended soft sets and combined them with fuzzy sets and rough sets, Aktas and Cagman [6] studied soft groups, and Acar et al. [7] presented soft rings. Polygroups were studied by Comer [8, 9]. Abbasizadeh and Davvaz [10] presented the concept of fuzzy topological polygroups and proved some results. Soft topological polygroups were based on work by Hidari et al. [11]. Recently, in [12], the author introduced the notion of soft topological polygroups by applying soft set theory to topological polygroups. In addition, see [13–16].

2. Preliminaries

Let U be an initial universe, and $\mathcal{P}(U)$ denote the power set of U . Suppose that E is a set of parameters, $A \subseteq E$, and $A \neq \emptyset$. A pair $(\mathbb{F}, A)$ is said to be a soft set over U , where $\mathbb{F} : A \rightarrow \mathcal{P}(U)$ denotes a map. Furthermore, a soft set over U is a parameterized family of subsets of universe U . For $a \in A$, $\mathbb{F}(a)$ may be considered as the set of approximate elements of the soft set $(\mathbb{F}, A)$. Note that a soft set is not a set [6, 7]. Suppose that $(\mathbb{F}, A)$ and $(\mathbb{G}, B)$ are soft sets over U . In this case, we say that $(\mathbb{F}, A)$ is a soft subset of $(\mathbb{G}, B)$ (i.e., $(\mathbb{F}, A) \hat{\subseteq} (\mathbb{G}, B)$) if $A \subseteq B$ and $\mathbb{F}(a) \subseteq \mathbb{G}(a)$ for all $a \in A$. If $(\mathbb{G}, B) \hat{\subseteq} (\mathbb{F}, A)$, then we say that $(\mathbb{F}, A)$ is a soft superset of $(\mathbb{G}, B)$, which is denoted by $(\mathbb{F}, A) \hat{\supseteq} (\mathbb{G}, B)$. Soft sets $(\mathbb{F}, A)$ and $(\mathbb{G}, B)$ are said to be soft equal and are denoted by $(\mathbb{F}, A) \hat{=} (\mathbb{G}, B)$ if $(\mathbb{F}, A) \hat{\subseteq} (\mathbb{G}, B)$ and $(\mathbb{G}, B) \hat{\subseteq} (\mathbb{F}, A)$.

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If $\mathbb{F}(a) = U$ for all $a \in A$, then $(\mathbb{F}, A)$ is an absolute soft set, denoted by $\hat{A}$. Otherwise, if $\mathbb{F}(a) = \emptyset$ (null set) for all $a \in A$, then $(\mathbb{F}, A)$ is a null soft set, denoted by $\hat{\emptyset}$. Symbol $(\mathbb{F}, A) \hat{\cap} (\mathbb{G}, B)$ denotes the bi-intersection of two soft sets, which is defined as $(\mathbb{F}, A) \hat{\cap} (\mathbb{G}, B) = (\mathbb{H}, C)$, where $C = A \cap B$ and $\mathbb{H}(a) = \mathbb{F}(a) \cap \mathbb{G}(a)$ for all $a \in C$. The soft set $(\mathbb{H}, C)$ defines a union of $(\mathbb{F}, A)$ and $(\mathbb{G}, B)$, which is shown by $(\mathbb{F}, A) \hat{\cup} (\mathbb{G}, B) = (\mathbb{H}, C)$, where $C = A \cup B$ and

$$\mathbb{H}(a) = \begin{cases} \mathbb{F}(a), & \text{if } a \in A - B; \\ \mathbb{G}(a), & \text{if } a \in B - A; \\ \mathbb{F}(a) \cup \mathbb{G}(a), & \text{if } a \in A \cap B. \end{cases}$$

In addition, $(\mathbb{F}, A)$ AND $(\mathbb{G}, B)$ is denoted by $(\mathbb{F}, A) \hat{\wedge} (\mathbb{G}, B)$ and defined by $(\mathbb{F}, A) \hat{\wedge} (\mathbb{G}, B) = (\mathbb{H}, A \times B)$, where $\mathbb{H}((a, b)) = \mathbb{F}(a) \cap \mathbb{G}(b)$ for all $(a, b) \in A \times B$. Similarly, $(\mathbb{F}, A)$ OR $(\mathbb{G}, B)$ is denoted by $(\mathbb{F}, A) \hat{\vee} (\mathbb{G}, B)$ and defined by $(\mathbb{F}, A) \hat{\vee} (\mathbb{G}, B) = (\mathbb{I}, A \times B)$, where $\mathbb{I}((a, b)) = \mathbb{F}(a) \cup \mathbb{G}(b)$ for all $(a, b) \in A \times B$. The support of the soft set $(\mathbb{F}, A)$ is defined as $Supp(\mathbb{F}, A) = \{a \in A : \mathbb{F}(a) \neq \emptyset\}$. If the support of the soft set $(\mathbb{F}, A)$ is not equal to the empty set, we say that $(\mathbb{F}, A)$ is non-null.

Let H be a non-empty set. Pair $(H, \circ)$ is called a hypergroupoid if $\circ : H \times H \rightarrow \mathcal{P}(H)$ is a map, where $\mathcal{P}(H)$ is a family of non-empty subsets of H . A hypergroupoid $(H, \circ)$ is called a quasihypergroup if, for every $h \in H$, we have $h \circ H = H = H \circ h$, and it is called a semihypergroup if for every $t, u, w \in H$, we have $t \circ (u \circ w) = (t \circ u) \circ w$. Pair $(H, \circ)$ is called a hypergroup if it is a quasihypergroup and a semi-hypergroup. A special type of hypergroups is a polygroup. A hypergroup $P = (P, \circ, e, {}^{-1})$ is a polygroup if there exists a unitary operation ${}^{-1}$ on P and $e \in P$ such that for all $p, q, r \in P$, the following conditions hold: (1) $(p \circ q) \circ r = p \circ (q \circ r)$; (2) $e \circ p = p \circ e = p$; (3) if $p \in q \circ r$, then $q \in p \circ r^{-1}$ and $r \in q^{-1} \circ p$. The following results about polygroups follow easily from the definition: $e \in p \circ p^{-1} \cap p^{-1} \circ p$, $e^{-1} = e$, $(p^{-1})^{-1} = p$, and $(p \circ q)^{-1} = q^{-1} \circ p^{-1}$ [9, 17]. Let P be a polygroup and $(\mathbb{F}, A)$ be a non-null soft set over P . Then, $(\mathbb{F}, A)$ is called a (normal) soft polygroup over P if $\mathbb{F}(x)$ is a (normal) subpolygroup of P for all $x \in Supp(\mathbb{F}, A)$.

3. Topological Hyperstructure

A topological group is a group G with a topology on G that satisfies the following two properties:

- (1) Mapping $p : G \times G \rightarrow G$ defined by $p(g, h) = gh$ is

continuous when $G \times G$ is endowed with the product topology;

- (2) Mapping $inv : G \rightarrow G$ defined by $inv(g) = g^{-1}$ is continuous [18].

Let (F, A) be a non-null soft set defined over G . Then, triplet (F, A, τ) is called a soft topological group over G if the following conditions are satisfied:

- (a) $F(a)$ is a subgroup of G for all $a \in A$;
- (b) Mapping $(x, y) \mapsto xy$ of the topological space $F(a) \times F(a)$ to $F(a)$ and mapping $x \mapsto x^{-1}$ of the topological space $F(a)$ to $F(a)$ are continuous for all $a \in A$.

Example 1. Take $G = S_3 = \{e, (12), (13), (23), (123), (132)\}$, $A = \{e_1, e_2, e_3\}$, and let the base for topology τ be $B = \{\{e\}, \{(12)\}, \{(123)\}, \{(132)\}\}$. The set-valued function F is defined by $F(e_1) = \{e\}$, $F(e_2) = \{e, (12)\}$, and $F(e_3) = \{e, (123), (132)\}$. Clearly, $F(a)$ is a subgroup of G , for all $a \in A$. In addition, condition (b) was satisfied. Hence, (F, A, τ) is a soft topological group over G .

Let (H, τ) be a topological space. The following theorem provides a topology on $\mathcal{P}^*(H)$ that is induced by τ .

Theorem 1 ([19]). *Let (H, τ) be a topological space. Then, family β consisting of all sets $S_V = \{U \in \mathcal{P}^*(H) \mid U \subseteq V, V \in \tau\}$ is a base for a topology on $\mathcal{P}^*(H)$. This topology is denoted by τ^* .*

Let $(H, \circ)$ be a hypergroup and (H, τ) be a topological space. Then, system $(H, \circ, \tau)$ is called a topological hypergroup if the following conditions are satisfied: (1) mapping $(x, y) \mapsto x \circ y$, from $H \times H \rightarrow \mathcal{P}^*(H)$, is continuous; (2) mapping $(x, y) \mapsto x/y$, from $H \times H \rightarrow \mathcal{P}^*(H)$, is continuous, where $x/y = \{z \in H \mid x \in z \circ y\}$; (3) mapping $(x, y) \mapsto y \backslash x$, from $H \times H$ to $\mathcal{P}^*(H)$, is continuous, where $y \backslash x = \{z \in H \mid x \in y \circ z\}$. Let $\langle P, \circ, e, {}^{-1} \rangle$ be a polygroup and (P, τ) be a topological space. Then, system $\langle P, \circ, e, {}^{-1}, \tau \rangle$ is called a topological polygroup if mappings $\circ : P \times P \rightarrow \mathcal{P}^*(P)$ and ${}^{-1} : P \rightarrow P$ are continuous. Obviously, every topological group is a topological polygroup. The following theorem helps us to recognize continuous hyperoperations.

Theorem 2 ([11]). *Let P be a polygroup. Then, hyperoperation $\circ : P \times P \rightarrow \mathcal{P}^*(P)$ is continuous if and only if for every $x, y \in P$ and $U \in \tau$ such that $x \circ y \subseteq U$ there exist $V, W \in \tau$ such that $x \in V$ and $y \in W$ and $V \circ W \subseteq U$.*

Example 2 ([11]). Let P be a polygroup and β^* be the fundamental relation of P . Then, $\tau = \{\bigcup_{u \in U} \beta^*(u) \mid U \subseteq P\} \cup \{\emptyset\}$ is a topology on P , and $(P, \circ, e, {}^{-1}, \tau)$ is a topological polygroup.

4. Soft Topological Polygroups

In [12], the author defined the concept of soft topological polygroups and achieved several results by establishing important characterizations of this concept. We recall the definition of soft topological polygroups as follows.

Definition 1 ([12]). Let Θ be a topology defined on a polygroup P . Let $(\mathbb{F}, A)$ be a soft set defined over P . Then, system $(\mathbb{F}, A, \Theta)$ is called a soft topological polygroup over P if (a) $\mathbb{F}(a)$ is a subgroup of P for all $a \in A$, and (b) mapping $(x, y) \mapsto x \circ (-y)$ of the topological space $\mathbb{F}(a) \times \mathbb{F}(a)$ onto $\mathcal{P}(\mathbb{F}(a))$ is continuous for all $a \in A$.

Topology Θ on P induces topologies on $\mathbb{F}(a)$, $\mathbb{F}(a) \times \mathbb{F}(a)$, and $\mathcal{P}(\mathbb{F}(a))$.

Example 3. Let P be a polygroup, and let Θ be a discrete or anti-discrete topology. Suppose that $(\mathbb{F}, A)$ is a soft polygroup on P ; in this case, $(\mathbb{F}, A, \Theta)$ is a soft topological polygroup.

It is easy to verify that any soft polygroup satisfies condition (b) of Definition 1 with both topologies.

Example 4. Every soft topological group is a soft topological polygroup.

Example 5. Let (P, Θ) be a topological polygroup, and let all subpolygroups of P be $H_1, H_2, \dots, H_n$. Let A be an arbitrary set such that $a_1, a_2, \dots, a_n \in A$. We define $\mathbb{F}$ as follows:

$$\mathbb{F}(x) = \begin{cases} H_i, & \text{if } x = a_i \ (i = 1, \dots, n); \\ \emptyset, & \text{otherwise.} \end{cases}$$

With the above conditions, $(\mathbb{F}, A, \Theta)$ is a soft topological polygroup over P . In fact, every topological polygroup becomes a soft topological polygroup using this method.

Example 6. Suppose that (G, Θ) is a topological group, where H is a compact subgroup of G . It is well-known that system $G//H = (\{HgH, g \in G\}, \circ, H, -1)$ is a polygroup with hyperoperation $(HaH) \circ (HbH) = \{HahbH \mid h \in H\}$ and the unitary function $(HgH)^{-1} = \{Hg^{-1}H\}$. Let $\pi : G \mapsto G//H$ be a function, where $\pi(g) = HgH$. We define topology $\Theta_{G//H}$ on $G//H$ as follows. A subset U of $G//H$ is open if $\pi^{-1}(U)$ is an open subset of G . Therefore, $(G//H, \Theta_{G//H})$ is a topological polygroup. Now, if $H_1, H_2, \dots, H_n$ are subpolygroups of $(G//H, \Theta_{G//H})$, then, using the method from Example 5, we can construct a soft topological polygroup $(\mathbb{F}, A, \Theta_{G//H})$, where A is an arbitrary non-empty subset of the parameter set E .

Theorem 3. Every soft polygroup over a topological polygroup (anti-discrete) is a soft topological polygroup.

Proof. Suppose that $(\mathbb{F}, A)$ is a soft polygroup over P and (P, Θ) is a topological polygroup. Therefore, for all $a \in A$, $\mathbb{F}(a)$ is a subpolygroup of P . Mapping $\mathbb{F}(a) \times \mathbb{F}(a)$ to $\mathcal{P}(\mathbb{F}(a))$ is continuous. Note that P is a topological polygroup, and mapping $(a, b) \mapsto a \circ b^{-1}$ of the topological space $P \times P$ to $\mathcal{P}(P)$ is continuous. Hence, $(\mathbb{F}, A, \Theta)$ is a soft topological polygroup over P . $\square$

Example 7. Let $P = \{1, 2\}$, and let hyperoperation $*$ be defined by $1*1 = 1, 1*2 = 2*1 = 2$, and $2*2 = \{1, 2\}$. Let Θ_1 be topology $\{\emptyset, P, \{1\}\}$. By Theorem 2, hyperoperation $*$: $P \times P \mapsto \mathcal{P}(P)$ is continuous, and the inverse operation $^{-1} : P \mapsto P$ is continuous because $^{-1}$ is the identity function ($x^{-1} = x$ for all $x \in P$), and identity is continuous with every topology. However, hyperoperation $*$: $P \times P \mapsto \mathcal{P}(P)$ is not continuous with $\Theta_2 = \{\emptyset, P, \{2\}\}$. Therefore, P with $\Theta_1, \Theta_{dis}, \Theta_{ndis}$ is a topological polygroup. Subpolygroups of P are $\emptyset, P, \{1\}$. Let A be an arbitrary set, and let $a_1, a_2 \in A$. We define the soft set $\mathbb{F}$ as follows:

$$\mathbb{F}(x) = \begin{cases} \{1\}, & \text{if } x = a_1; \\ P, & \text{if } x = a_2; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Therefore, $(\mathbb{F}, A, \Theta_1)$ is a soft topological polygroup.

Example 8. Let P be $\{e, a, b\}$ and the multiplication table be

$\circ$	e	a	b
e	e	a	b
a	a	e	b
b	b	b	$\{e, a\}$

By Theorem 2, hyperoperation $\circ : P \times P \mapsto \mathcal{P}(P)$ is not continuous with topologies $\Theta_1 = \{\emptyset, P, \{e\}\}$, $\Theta_2 = \{\emptyset, P, \{a\}\}$, $\Theta_3 = \{\emptyset, P, \{b\}\}$, $\Theta_4 = \{\emptyset, P, \{e, a\}\}$, $\Theta_5 = \{\emptyset, P, \{e, b\}\}$, and $\Theta_6 = \{\emptyset, P, \{a, b\}\}$. Subpolygroups of P are $\emptyset, P, \{e\}, \{e, a\}$. Let A be an arbitrary set, and $a_1, a_2 \in A$. We define a soft set $\mathbb{F}$ by

$$\mathbb{F}(x) = \begin{cases} \{e\}, & \text{if } x = a_1; \\ \{e, a\}, & \text{if } x = a_2; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Therefore, $(\mathbb{F}, A, \Theta_3), (\mathbb{F}, A, \Theta_4)$ is a soft topological polygroup because the restriction of topologies Θ_3, Θ_4 to subspaces $\{e\}, \{e, a\}$ are discrete or anti-discrete topologies.

Example 9. Let P be $\{e, a, b, c\}$ and the multiplication table be

$\circ$	e	a	b	c
e	e	a	b	c
a	a	$\{e, a\}$	c	$\{b, c\}$
b	b	c	e	a
c	c	$\{b, c\}$	a	$\{e, a\}$

Hyperoperation $\circ : P \times P \mapsto \mathcal{P}(P)$ is continuous with the following topologies: $\Theta_{dis}, \Theta_{ndis}, \Theta_1 = \{\emptyset, P, \{e, b\}\}, \Theta_2 = \{\emptyset, P, \{e\}, \{b\}\}$. As $x^{-1} = x$ for all $x \in P$, the inverse operation is identity, and the identity function is continuous with every topology. This means that P with topologies Θ_1, Θ_2 is a topological polygroup. Hyperoperation $\circ : P \times P \mapsto \mathcal{P}(P)$ with the following topologies is not continuous: $\Theta_3 = \{\emptyset, P, \{e\}\}, \Theta_4 = \{\emptyset, P, \{a\}\}, \Theta_5 = \{\emptyset, P, \{b\}\}, \Theta_6 = \{\emptyset, P, \{c\}\}, \Theta_7 = \{\emptyset, P, \{e, a\}\}, \Theta_8 = \{\emptyset, P, \{e, c\}\}, \Theta_9 = \{\emptyset, P, \{a, b\}\}, \Theta_{10} = \{\emptyset, P, \{a, c\}\}, \Theta_{11} = \{\emptyset, P, \{b, c\}\}, \Theta_{12} = \{\emptyset, P, \{e, a, b\}\}, \Theta_{13} = \{\emptyset, P, \{e, a, c\}\}, \Theta_{14} = \{\emptyset, P, \{e, b, c\}\}, \Theta_{15} = \{\emptyset, P, \{a, b, c\}\}, \Theta_{16} = \{\emptyset, P, \{e\}, \{a\}\}$. Subpolygroups of P are $\emptyset, P, \{e\}, \{e, a\}, \{e, b\}$. Let A be an arbitrary set, and $a_1, a_2, a_3, a_4 \in A$. We define the soft set $\mathbb{F}$:

$$\mathbb{F}(x) = \begin{cases} \{e\}, & \text{if } x = a_1; \\ \{e, a\}, & \text{if } x = a_2; \\ \{e, b\}, & \text{if } x = a_3; \\ P, & \text{if } x = a_4; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Therefore, $(\mathbb{F}, A, \Theta_1), (\mathbb{F}, A, \Theta_2)$ is a soft topological polygroup. Now, we consider $\Theta_5 = \{\emptyset, P, \{b\}\}$ and define a soft set $\mathbb{F}$:

$$\mathbb{F}(x) = \begin{cases} \{e\}, & \text{if } x = a_1; \\ \{e, a\}, & \text{if } x = a_2; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Hyperoperation $\circ : \mathbb{F}(x) \times \mathbb{F}(x) \mapsto \mathcal{P}(\mathbb{F}(x))$ is continuous with Θ_5 . Therefore, $(\mathbb{F}, A, \Theta_5)$ is a soft topological polygroup. Suppose that $\Theta_6 = \{\emptyset, P, \{c\}\}$. We define a soft set $\mathbb{F}$ as follows:

$$\mathbb{F}(x) = \begin{cases} \{e\}, & \text{if } x = a_1; \\ \{e, a\}, & \text{if } x = a_2; \\ \{e, b\}, & \text{if } x = a_3; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Hyperoperation $\circ : \mathbb{F}(x) \times \mathbb{F}(x) \mapsto \mathcal{P}(\mathbb{F}(x))$ is continuous with Θ_6 . Hence, $(\mathbb{F}, A, \Theta_6)$ is a soft topological polygroup. We can construct additional examples using this method.

Example 10. Consider the non-abelian polygroup $P = \{e, a, b, c\}$ with the following multiplication table:

$\circ$	e	a	b	c
e	e	a	b	c
a	a	a	P	c
b	b	$\{e, a, b\}$	b	$\{b, c\}$
c	c	$\{a, c\}$	c	P

By Theorem 2, hyperoperation $\circ : P \times P \mapsto \mathcal{P}(P)$ is continuous with the following topologies: $\Theta_{dis}, \Theta_{ndis}, \Theta_1 = \{\emptyset, P, \{e\}\}, \Theta_2 = \{\emptyset, P, \{e, a\}\}, \Theta_3 = \{\emptyset, P, \{e, b\}\}, \Theta_4 = \{\emptyset, P, \{e\}, \{a\}\}, \Theta_5 = \{\emptyset, P, \{e\}, \{b\}\}$. The inverse operation $^{-1}$ is continuous if the inverse of any open set is open. The inverse operation $^{-1}$ is not continuous with the topologies $\Theta_2, \Theta_3, \Theta_4, \Theta_5$ and continuous with Θ_1 . This means that P with topologies Θ_1, Θ_{dis} , and Θ_{ndis} is a topological polygroup. Hyperoperation $\circ : P \times P \mapsto \mathcal{P}(P)$ with the following topologies is not continuous: $\Theta_6 = \{\emptyset, P, \{a\}\}, \Theta_7 = \{\emptyset, P, \{b\}\}, \Theta_8 = \{\emptyset, P, \{c\}\}, \Theta_9 = \{\emptyset, P, \{e, c\}\}, \Theta_{10} = \{\emptyset, P, \{a, b\}\}, \Theta_{11} = \{\emptyset, P, \{a, c\}\}, \Theta_{12} = \{\emptyset, P, \{b, c\}\}, \Theta_{13} = \{\emptyset, P, \{e, a, b\}\}, \Theta_{14} = \{\emptyset, P, \{e, a, c\}\}, \Theta_{15} = \{\emptyset, P, \{e, b, c\}\}, \Theta_{16} = \{\emptyset, P, \{a, b, c\}\}$. Subpolygroups of P are $\emptyset, P, \{e\}$. Let A be an arbitrary set, and $a_1, a_2 \in A$. We define a soft set $\mathbb{F}$:

$$\mathbb{F}(x) = \begin{cases} \{e\}, & \text{if } x = a_1; \\ P, & \text{if } x = a_2; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Therefore, $(\mathbb{F}, A, \Theta_1), (\mathbb{F}, A, \Theta_{dis}), (\mathbb{F}, A, \Theta_{ndis})$ are soft topological polygroups. We define a soft set $\mathbb{F}$ as follows:

$$\mathbb{F}(x) = \begin{cases} \{e\}, & \text{if } x = a_1; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Hyperoperation $\circ : \mathbb{F}(x) \times \mathbb{F}(x) \mapsto \mathcal{P}(\mathbb{F}(x))$ and the inverse operation $^{-1}$ are continuous with $(\Theta_i)_{i=1, \dots, 16}$. Therefore, $(\mathbb{F}, A, \Theta_i)$ is a soft topological polygroup.

Using Theorems 4–9, we can build many other examples using the examples given in this article.

Theorem 4. Every soft polygroup over a topological polygroup (non-discrete) is a soft topological polygroup.

Proof. Suppose that (P, Θ) is a topological polygroup, and $(\mathbb{F}, A)$ is a soft polygroup over P . In this case, for all $a \in A$, $\mathbb{F}(a)$ is a subpolygroup of P . By contrast, P is a topological polygroup, and the mapping $(a, b) \mapsto a \circ b^{-1}$ of the topological space $P \times P$ to P is continuous. Thus, its restriction from $\mathbb{F}(a) \times \mathbb{F}(a)$ to $\mathbb{F}(a)$ is also continuous. Therefore, $(\mathbb{F}, A, \Theta)$ is a soft topological polygroup over (P, Θ) . $\square$

Theorem 5. Suppose that $(\mathbb{F}, A, \Theta)$ and $(\mathbb{G}, B, \Theta)$ are soft topological polygroups over P .

(1) $(\mathbb{F}, A, \Theta) \hat{\cap} (\mathbb{G}, B, \Theta)$ is a soft topological polygroup over P .

(2) $(\mathbb{F}, A, \Theta) \cap_E (\mathbb{G}, B, \Theta)$ is a soft topological polygroup over P .

Proof. (1) Note that $(\mathbb{F}, A, \Theta)$ and $(\mathbb{G}, B, \Theta)$ are soft topological polygroups over P . Therefore, their bi-intersection over P is the soft topological set $(\mathbb{H}, C, \Theta)$, where $C = A \cap B$. For all $c \in C$, we have $\mathbb{H}(c) = \mathbb{F}(c) \cap \mathbb{G}(c)$. In addition, both $\mathbb{F}(c)$ and $\mathbb{G}(c)$ are subpolygroups. Thus, $\mathbb{H}(c)$ is a subpolygroup of P for all $c \in A \cap B$. In contrast, $\mathbb{H}(c) \subseteq \mathbb{F}(c)$ and $\mathbb{G}(c)$ and condition (b) of Definition 1 hold for $\mathbb{F}(c)$ and $\mathbb{G}(c)$. Thus, they also hold for $\mathbb{H}(c)$ for all $c \in C$. Hence, $(\mathbb{F}, A, \Theta) \hat{\cap} (\mathbb{G}, B, \Theta)$ is a soft topological polygroup over P .

(2) $\mathbb{F}(c)$, $\mathbb{G}(c)$, and $\mathbb{F}(c) \cap \mathbb{G}(c)$ are subpolygroups of P , and condition (b) of Definition 1 is established for $\mathbb{F}(c)$, $\mathbb{G}(c)$, and $\mathbb{F}(c) \cap \mathbb{G}(c)$. Thus, it is also established for $\mathbb{H}(c)$ for all $c \in C$, where $C = A \cup B$. $\square$

Theorem 6. Bi-intersection $\hat{\cap}_{i \in I} (\mathbb{F}_i, A_i, \Theta)$ is a soft topological polygroup over P , where $(\mathbb{F}_i, A_i, \Theta), i \in I$ is a nonempty family of soft topological polygroups over P .

Proof. Suppose that $C = \cap_{i \in I} A_i$ and $\mathbb{H}(c) = \cap_{i \in I} \mathbb{F}_i(c)$. We have $\mathbb{F}_i(c)$ that are subpolygroups of P . Therefore, $\mathbb{H}(c) = \cap_{i \in I} \mathbb{F}_i(c)$ are subpolygroups of P as well, and condition (b) of Definition 1 holds for $\mathbb{F}_i(c)$ and is also established for $\mathbb{H}(c) = \cap_{i \in I} \mathbb{F}_i(c)$. $\square$

Theorem 7. Let Θ be a topology defined over P , and let $(\mathbb{F}, A, \Theta)$ and $(\mathbb{G}, B, \Theta)$ be soft topological polygroups over P . The following statements are true:

(1) $(\mathbb{F}, A, \Theta) \hat{\cap} (\mathbb{G}, B, \Theta)$ is a soft topological polygroup over P .

(2) $(\mathbb{F}, A, \Theta) \hat{\cup} (\mathbb{G}, B, \Theta)$ is a soft topological polygroup over P , if A and B are disjoint.

Proof. (1) If $C = A \times B$, $\mathbb{H}(a, b) = \mathbb{F}(a) \cap \mathbb{G}(b)$, then $\mathbb{H}(a, b)$ is a subpolygroup of P . Condition (b) of Definition 1 holds on

$\mathbb{F}(a)$. Thus, $\mathbb{G}(b)$ is established on $\mathbb{F}(a) \cap \mathbb{G}(b)$.

(2) If $C = A \cup B$, $\mathbb{H}(c) = \mathbb{F}(c)$ or $\mathbb{G}(c)$, then it is clear that conditions (a) and (b) of Definition 1 are established on $\mathbb{H}(c)$. $\square$

Theorem 8. Suppose that $(\mathbb{F}_i, A_i, \Theta), i \in I$ is a nonempty family of soft topological polygroups over P . Then, the following statements hold:

(1) $\hat{\cap}_{i \in I} (\mathbb{F}_i, A_i, \Theta)$ is a soft topological polygroup over P .

(2) If A_i is disjoint, $\hat{\cup}_{i \in I} (\mathbb{F}_i, A_i, \Theta)$ is a soft topological polygroup over P .

Proof. The proof is straightforward. $\square$

Example 11. By referring to [9, 17], we can construct polygroup $\overline{D}_4$ as follows:

$\circ$	C_1	C_2	C_3	C_4	C_5
C_1	C_1	C_2	C_3	C_4	C_5
C_2	C_2	C_1	C_3	C_4	C_5
C_3	C_3	C_3	C_1, C_2	C_5	C_4
C_4	C_4	C_4	C_5	C_1, C_2	C_3
C_5	C_5	C_5	C_4	C_3	C_1, C_2

As a sample of how to calculate the table entries, consider $C_3 \cdot C_3$. To determine this product, we compute the element-wise product of the conjugacy classes $\{r, t\}\{r, t\} = \{s, 1\} = C_1 \cup C_2$. Thus, $C_3 \circ C_3$ consists of the two conjugacy classes C_1, C_2 . Hyperoperation $\circ : \overline{D}_4 \times \overline{D}_4 \mapsto \mathcal{P}(\overline{D}_4)$ is not continuous with the following topologies: $\Theta_1 = \{\emptyset, \overline{D}_4, \{C_1\}\}$, $\Theta_2 = \{\emptyset, \overline{D}_4, \{C_2\}\}$, $\Theta_3 = \{\emptyset, \overline{D}_4, \{C_3\}\}$, $\Theta_4 = \{\emptyset, \overline{D}_4, \{C_4\}\}$, $\Theta_5 = \{\emptyset, \overline{D}_4, \{C_5\}\}$, $\Theta_6 = \{\emptyset, \overline{D}_4, \{C_1, C_2\}\}$, $\Theta_7 = \{\emptyset, \overline{D}_4, \{C_1, C_3\}\}$, $\Theta_8 = \{\emptyset, \overline{D}_4, \{C_1, C_4\}\}$, $\Theta_9 = \{\emptyset, \overline{D}_4, \{C_1, C_5\}\}$, $\Theta_{10} = \{\emptyset, \overline{D}_4, \{C_2, C_3\}\}$, $\Theta_{11} = \{\emptyset, \overline{D}_4, \{C_2, C_4\}\}$, $\Theta_{12} = \{\emptyset, \overline{D}_4, \{C_2, C_5\}\}$, $\Theta_{13} = \{\emptyset, \overline{D}_4, \{C_3, C_4\}\}$, $\Theta_{14} = \{\emptyset, \overline{D}_4, \{C_3, C_5\}\}$, $\Theta_{15} = \{\emptyset, \overline{D}_4, \{C_4, C_5\}\}$, $\Theta_{16} = \{\emptyset, \overline{D}_4, \{C_1, C_2, C_3\}\}$, $\Theta_{17} = \{\emptyset, \overline{D}_4, \{C_1, C_2, C_4\}\}$, $\Theta_{18} = \{\emptyset, \overline{D}_4, \{C_1, C_2, C_5\}\}$, $\Theta_{19} = \{\emptyset, \overline{D}_4, \{C_2, C_3, C_4\}\}$, $\Theta_{20} = \{\emptyset, \overline{D}_4, \{C_2, C_3, C_5\}\}$, $\Theta_{21} = \{\emptyset, \overline{D}_4, \{C_3, C_4, C_5\}\}$, $\Theta_{22} = \{\emptyset, \overline{D}_4, \{C_1, C_2, C_3, C_4\}\}$, $\Theta_{23} = \{\emptyset, \overline{D}_4, \{C_1, C_2, C_3, C_5\}\}$, $\Theta_{24} = \{\emptyset, \overline{D}_4, \{C_2, C_3, C_4, C_5\}\}$. This means that $(\overline{D}_4, \Theta_{dis})$ and $(\overline{D}_4, \Theta_{ndis})$ are topological polygroups. Subpolygroups of $\overline{D}_4$ are $\emptyset, \overline{D}_4, \{C_1\}, \{C_1, C_2\}, \{C_1, C_2, C_3\}, \{C_1, C_2, C_4\}, \{C_1, C_2, C_5\}$. Let A be an arbitrary set, and $a_1, a_2, a_3, a_4 \in A$. We define a soft set $\mathbb{F}$ by

$$\mathbb{F}(x) = \begin{cases} \{C_1\}, & \text{if } x = a_1; \\ \{C_1, C_2\}, & \text{if } x = a_2; \\ \{C_1, C_2, C_3\}, & \text{if } x = a_3; \\ \{C_1, C_2, C_4\}, & \text{if } x = a_4; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Consider $\Theta_5 = \{\emptyset, \widehat{D_4}, \{C_5\}\}$. In this case, $(\mathbb{F}, A, \Theta_5)$ is a soft topological polygroup. This example is a good template for creating more polygroups and soft topological polygroups.

Example 12. By referring to [9, 17], we can construct polygroup $\widehat{D_4}$ as follows:

*	1	2	3	4	5
1	1	2	3	4	5
2	2	1	4	3	5
3	3	4	1	2	5
4	4	3	2	1	5
5	5	5	5	5	{1, 2, 3, 4}

Hyperoperation $*$: $\widehat{D_4} \times \widehat{D_4} \mapsto \mathcal{P}(\widehat{D_4})$ is not continuous with the following topologies: $\Theta_1 = \{\emptyset, \widehat{D_4}, \{1\}\}$, $\Theta_2 = \{\emptyset, \widehat{D_4}, \{2\}\}$, $\Theta_3 = \{\emptyset, \widehat{D_4}, \{3\}\}$, $\Theta_4 = \{\emptyset, \widehat{D_4}, \{4\}\}$, $\Theta_5 = \{\emptyset, \widehat{D_4}, \{5\}\}$, $\Theta_6 = \{\emptyset, \widehat{D_4}, \{1, 2\}\}$, $\Theta_7 = \{\emptyset, \widehat{D_4}, \{1, 3\}\}$, $\Theta_8 = \{\emptyset, \widehat{D_4}, \{1, 4\}\}$, $\Theta_9 = \{\emptyset, \widehat{D_4}, \{1, 5\}\}$, $\Theta_{10} = \{\emptyset, \widehat{D_4}, \{2, 3\}\}$, $\Theta_{11} = \{\emptyset, \widehat{D_4}, \{2, 4\}\}$, $\Theta_{12} = \{\emptyset, \widehat{D_4}, \{2, 5\}\}$, $\Theta_{13} = \{\emptyset, \widehat{D_4}, \{3, 4\}\}$, $\Theta_{14} = \{\emptyset, \widehat{D_4}, \{3, 5\}\}$, $\Theta_{15} = \{\emptyset, \widehat{D_4}, \{4, 5\}\}$, $\Theta_{16} = \{\emptyset, \widehat{D_4}, \{1, 2, 3\}\}$, $\Theta_{17} = \{\emptyset, \widehat{D_4}, \{1, 2, 4\}\}$, $\Theta_{18} = \{\emptyset, \widehat{D_4}, \{1, 2, 5\}\}$, $\Theta_{19} = \{\emptyset, \widehat{D_4}, \{2, 3, 4\}\}$, $\Theta_{20} = \{\emptyset, \widehat{D_4}, \{2, 3, 5\}\}$, $\Theta_{21} = \{\emptyset, \widehat{D_4}, \{2, 4, 5\}\}$, $\Theta_{22} = \{\emptyset, \widehat{D_4}, \{3, 4, 5\}\}$, $\Theta_{23} = \{\emptyset, \widehat{D_4}, \{1, 2, 3, 4\}\}$, $\Theta_{24} = \{\emptyset, \widehat{D_4}, \{1, 2, 3, 5\}\}$, $\Theta_{25} = \{\emptyset, \widehat{D_4}, \{2, 3, 4, 5\}\}$. This means that $(\widehat{D_4}, \Theta_{dis})$, and $(\widehat{D_4}, \Theta_{ndis})$ are topological polygroups. Subpolygroups of $\widehat{D_4}$ are $\emptyset, \widehat{D_4}, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3, 4\}$. Let A be an arbitrary set, and let $a_1, a_2, a_3, a_4, a_5 \in A$. Then, we define a soft set $\mathbb{F}$ as follows:

$$\mathbb{F}(x) = \begin{cases} \{1\}, & \text{if } x = a_1; \\ \{1, 2\}, & \text{if } x = a_2; \\ \{1, 3\}, & \text{if } x = a_3; \\ \{1, 4\}, & \text{if } x = a_4; \\ \{1, 2, 3, 4\}, & \text{if } x = a_5; \\ \emptyset, & \text{otherwise.} \end{cases}$$

If we consider $\Theta_5 = \{\emptyset, \widehat{D_4}, \{5\}\}$, then $(\mathbb{F}, A, \Theta_5)$ is a soft topological polygroup. This horizontal approach creates an opportunity for us to build more examples.

Definition 2. Let $(\mathbb{F}_1, A_1, \Theta_1)$ on P_1 and $(\mathbb{F}_2, A_2, \Theta_2)$ on P_2

be two soft topological polygroups. The product of $(\mathbb{F}_1, A_1, \Theta_1)$ and $(\mathbb{F}_2, A_2, \Theta_2)$ is denoted by $(\mathbb{F}_1, A_1, \Theta_1) \times (\mathbb{F}_2, A_2, \Theta_2) = (\mathbb{F}_1 \times \mathbb{F}_2, A_1 \times A_2, \Theta_1 \times \Theta_2)$, where $\Theta_1 \times \Theta_2$ induce topology on $P_1 \times P_2$ and $\mathbb{F}_1 \times \mathbb{F}_2 : A_1 \times A_2 \mapsto \mathcal{P}(P_1 \times P_2)$, such that $(\mathbb{F}_1 \times \mathbb{F}_2)(a, b) = \mathbb{F}_1(a) \times \mathbb{F}_2(b)$, and $*$: $(\mathbb{F}_1 \times \mathbb{F}_2)(a, b) \times (\mathbb{F}_1 \times \mathbb{F}_2)(a, b) \mapsto \mathcal{P}(\mathbb{F}_1 \times \mathbb{F}_2)(a, b)$, where $(x, y) * (z, t) = (x \circ z, y \diamond t)$, such that $x \circ z \subseteq \mathbb{F}_1(a)$ and $y \diamond t \subseteq \mathbb{F}_2(b)$. On the other hand, $^{-1} : (\mathbb{F}_1 \times \mathbb{F}_2)(a, b) \mapsto (\mathbb{F}_1 \times \mathbb{F}_2)(a, b)$, where $(x, y) \rightsquigarrow (x^{-1}, y^{-1})$.

Theorem 9. A product of two soft topological polygroups is a soft topological polygroup.

Proof. Suppose that $(\mathbb{F}_1, A_1, \Theta_1)$ on P_1 and $(\mathbb{F}_2, A_2, \Theta_2)$ on P_2 are two soft topological polygroups. $P_1 \times P_2$ is a polygroup, and $\mathbb{F}_1(a) \times \mathbb{F}_2(b)$ is a subpolygroup of $P_1 \times P_2$. Maps $*$: $(\mathbb{F}_1 \times \mathbb{F}_2)(a, b) \times (\mathbb{F}_1 \times \mathbb{F}_2)(a, b) \mapsto \mathcal{P}(\mathbb{F}_1 \times \mathbb{F}_2)(a, b)$ and $^{-1}$: $(\mathbb{F}_1 \times \mathbb{F}_2)(a, b) \mapsto (\mathbb{F}_1 \times \mathbb{F}_2)(a, b)$ are continuous because maps $\circ : \mathbb{F}_1(a) \times \mathbb{F}_1(a) \mapsto \mathcal{P}(\mathbb{F}_1(a))$, $^{-1} : \mathbb{F}_1(a) \mapsto \mathbb{F}_1(a)$ and $\diamond : \mathbb{F}_2(b) \times \mathbb{F}_2(b) \mapsto \mathcal{P}(\mathbb{F}_2(b))$, $^{-1} : \mathbb{F}_2(b) \mapsto \mathbb{F}_2(b)$ are continuous. Thus, $(\mathbb{F}_1 \times \mathbb{F}_2, A_1 \times A_2, \Theta_1 \times \Theta_2)$ is a soft topological polygroup on $P_1 \times P_2$. $\square$

Extensions of polygroups by polygroups were investigated in [9]. By referring to [8, 17], we can construct $\mathcal{A}[\mathcal{B}]$. Several special cases of algebra $\mathcal{A}[\mathcal{B}]$ are useful. Before describing them, we need to assign names to the two 2-element polygroups. Let 2 denote group Z_2 , and let 3 denote polygroup $S_3/\langle(12)\rangle \cong Z_3/\theta$, where θ is the special conjugation with blocks $\{0\}, \{1, 2\}$. The multiplication table for 3 is as follows:

	0	1
0	0	2
1	1	{0, 1}

System $3[\mathcal{M}]$ is the result of adding a new identity to polygroup $[\mathcal{M}]$. System $2[\mathcal{M}]$ is almost as good. For example, suppose that $\mathcal{R}$ is the system with the following table:

	0	1	2
0	0	1	2
1	1	{0, 2}	{1, 2}
2	2	{1, 2}	{0, 1}

Example 13. Consider polygroup $2[\mathcal{R}]$:

$\circ$	0	a	1	2
0	0	a	1	2
a	a	0	1	2
1	1	1	{0, a, 2}	{1, 2}
2	2	2	{1, 2}	{0, a, 1}

Hyperoperation $\circ : 2[\mathcal{R}] \times 2[\mathcal{R}] \mapsto \mathcal{P}(2[\mathcal{R}])$ is not continuous with the following topologies: $\Theta_1 = \{\emptyset, 2[\mathcal{R}], \{0\}\}$, $\Theta_2 = \{\emptyset, 2[\mathcal{R}], \{a\}\}$, $\Theta_3 = \{\emptyset, 2[\mathcal{R}], \{1\}\}$, $\Theta_4 = \{\emptyset, 2[\mathcal{R}], \{2\}\}$, $\Theta_5 = \{\emptyset, 2[\mathcal{R}], \{0, 1\}\}$, $\Theta_6 = \{\emptyset, 2[\mathcal{R}], \{0, 2\}\}$, $\Theta_7 = \{\emptyset, 2[\mathcal{R}], \{a, 1\}\}$, $\Theta_8 = \{\emptyset, 2[\mathcal{R}], \{a, 2\}\}$, $\Theta_9 = \{\emptyset, 2[\mathcal{R}], \{1, 2\}\}$, $\Theta_{10} = \{\emptyset, 2[\mathcal{R}], \{0, a, 1\}\}$, $\Theta_{11} = \{\emptyset, 2[\mathcal{R}], \{0, a, 2\}\}$, $\Theta_{12} = \{\emptyset, 2[\mathcal{R}], \{a, 1, 2\}\}$, $\Theta_{13} = \{\emptyset, 2[\mathcal{R}], \{0, 1, 2\}\}$. However, $\circ : 2[\mathcal{R}] \times 2[\mathcal{R}] \mapsto \mathcal{P}(2[\mathcal{R}])$ is continuous with $\Theta_{14} = \{\emptyset, 2[\mathcal{R}], \{0, a\}\}$, $\Theta_{15} = \{\emptyset, 2[\mathcal{R}], \{0\}, \{a\}\}$. This means that $(2[\mathcal{R}], \Theta_{dis})$, $(2[\mathcal{R}], \Theta_{ndis})$, $(2[\mathcal{R}], \Theta_{14})$, and $(2[\mathcal{R}], \Theta_{15})$ are topological polygroups. Subpolygroups of $2[\mathcal{R}]$ are $\emptyset$, $2[\mathcal{R}]$, $\{0\}$, $\{0, a\}$. Let A be an arbitrary set, and let $a_1, a_2, a_3 \in A$. We define a soft set $\mathbb{F}$ by

$$\mathbb{F}(x) = \begin{cases} \{0\}, & \text{if } x = a_1; \\ \{0, a\}, & \text{if } x = a_2; \\ 2[\mathcal{R}], & \text{if } x = a_3; \\ \emptyset & \text{otherwise.} \end{cases}$$

In this case, $(\mathbb{F}, A, \Theta_{14})$ and $(\mathbb{F}, A, \Theta_{15})$ are soft topological polygroups.

Example 14. Consider polygroup $3[\mathcal{R}]$:

$\circ$	0	a	1	2
0	0	a	1	2
a	a	$\{0, a\}$	1	2
1	1	1	$\{0, a, 2\}$	$\{1, 2\}$
2	2	2	$\{1, 2\}$	$\{0, a, 1\}$

Hyperoperation $\circ : 3[\mathcal{R}] \times 3[\mathcal{R}] \mapsto \mathcal{P}(3[\mathcal{R}])$ is not continuous with the following topologies: $\Theta_1 = \{\emptyset, 3[\mathcal{R}], \{a\}\}$, $\Theta_2 = \{\emptyset, 3[\mathcal{R}], \{1\}\}$, $\Theta_3 = \{\emptyset, 3[\mathcal{R}], \{2\}\}$, $\Theta_4 = \{\emptyset, 3[\mathcal{R}], \{0, 1\}\}$, $\Theta_5 = \{\emptyset, 3[\mathcal{R}], \{0, 2\}\}$, $\Theta_6 = \{\emptyset, 3[\mathcal{R}], \{a, 1\}\}$, $\Theta_7 = \{\emptyset, 3[\mathcal{R}], \{a, 2\}\}$, $\Theta_8 = \{\emptyset, 3[\mathcal{R}], \{1, 2\}\}$, $\Theta_9 = \{\emptyset, 3[\mathcal{R}], \{0, a, 1\}\}$, $\Theta_{10} = \{\emptyset, 3[\mathcal{R}], \{0, a, 2\}\}$, $\Theta_{11} = \{\emptyset, 3[\mathcal{R}], \{a, 1, 2\}\}$. However, $\circ : 3[\mathcal{R}] \times 3[\mathcal{R}] \mapsto \mathcal{P}(3[\mathcal{R}])$ is continuous with $\Theta_{12} = \{\emptyset, 3[\mathcal{R}], \{0\}\}$, $\Theta_{13} = \{\emptyset, 3[\mathcal{R}], \{0, a\}\}$, $\Theta_{14} = \{\emptyset, 3[\mathcal{R}], \{0\}, \{a\}\}$. Therefore, $(3[\mathcal{R}], (\Theta_i)_{i=12,13,14})$ are topological polygroups. Subpolygroups of $3[\mathcal{R}]$ are $\emptyset$, $3[\mathcal{R}]$, $\{0\}$, and $\{0, a\}$. Let A be $3[\mathcal{R}]$. We define a soft set $\mathbb{F}$ as

$$\mathbb{F}(x) = \begin{cases} \{0\}, & \text{if } x = 0; \\ \{0, a\}, & \text{if } x = a; \\ 3[\mathcal{R}], & \text{if } x = 1; \\ \emptyset, & \text{if } x = 2. \end{cases}$$

Then, $(\mathbb{F}, A, (\Theta_i)_{i=12,13,14})$ is a soft topological polygroup. Now, let A be an arbitrary set, and $a_1, a_2 \in A$. We consider a soft set $\mathbb{F}$:

$$\mathbb{F}(x) = \begin{cases} \emptyset, & \text{if } x = a_1; \\ \{0, a\}, & \text{if } x = a_2; \\ \{0\}, & \text{otherwise.} \end{cases}$$

In this case, $(\mathbb{F}, A, (\Theta_i)_{i=3,4,5,8,9,10})$ are soft topological polygroups.

Example 15. Consider polygroup $\mathcal{R}[2]$:

$\circ$	0	1	2	a
0	0	1	2	a
1	1	$\{0, 2\}$	$\{1, 2\}$	a
2	2	$\{1, 2\}$	$\{0, 1\}$	a
a	a	a	a	$\{0, 1, 2\}$

Hyperoperation $\circ : \mathcal{R}[2] \times \mathcal{R}[2] \mapsto \mathcal{P}(\mathcal{R}[2])$ is not continuous with the following topologies: $\Theta_1 = \{\emptyset, \mathcal{R}[2], \{1\}\}$, $\Theta_2 = \{\emptyset, \mathcal{R}[2], \{2\}\}$, $\Theta_3 = \{\emptyset, \mathcal{R}[2], \{a\}\}$, $\Theta_4 = \{\emptyset, \mathcal{R}[2], \{0, 1\}\}$, $\Theta_5 = \{\emptyset, \mathcal{R}[2], \{0, 2\}\}$, $\Theta_6 = \{\emptyset, \mathcal{R}[2], \{0, a\}\}$, $\Theta_7 = \{\emptyset, \mathcal{R}[2], \{1, 2\}\}$, $\Theta_8 = \{\emptyset, \mathcal{R}[2], \{1, a\}\}$, $\Theta_9 = \{\emptyset, \mathcal{R}[2], \{2, a\}\}$, $\Theta_{10} = \{\emptyset, \mathcal{R}[2], \{0, 1, 2\}\}$, $\Theta_{11} = \{\emptyset, \mathcal{R}[2], \{0, 1, a\}\}$, $\Theta_{12} = \{\emptyset, \mathcal{R}[2], \{0, 2, a\}\}$. However, $\circ : \mathcal{R}[2] \times \mathcal{R}[2] \mapsto \mathcal{P}(\mathcal{R}[2])$ is continuous with $\Theta_{13} = \{\emptyset, \mathcal{R}[2], \{0\}\}$. Therefore, $(\mathcal{R}[2], \Theta_{dis})$, $(\mathcal{R}[2], \Theta_{ndis})$, and $(\mathcal{R}[2], \Theta_{13})$ are soft topological polygroups. Subpolygroups of $\mathcal{R}[2]$ are $\emptyset$, $\mathcal{R}[2]$, $\{0\}$, $\{0, 1, 2\}$.

Let A be an arbitrary set, and $a_1, a_2, a_3 \in A$. We define a soft set $\mathbb{F}$ by

$$\mathbb{F}(x) = \begin{cases} \{0\}, & \text{if } x = a_1; \\ \{0, 1, 2\}, & \text{if } x = a_2; \\ \emptyset, & \text{if } x = a_3; \\ \mathcal{R}[2], & \text{otherwise.} \end{cases}$$

Then, $(\mathbb{F}, A, \Theta_{13})$ is a soft topological polygroup. If $A = \mathcal{R}[2]$, we define a soft set $\mathbb{F}$ by

$$\mathbb{F}(x) = \begin{cases} \{0\}, & \text{if } x = a; \\ \{0, 1, 2\}, & \text{if } x = 2; \\ \emptyset, & \text{otherwise,} \end{cases}$$

then $(\mathbb{F}, A, (\Theta_i)_{i=3,6,10})$ is a soft topological polygroup.

Example 16. Consider polygroup $\mathcal{R}[3]$:

$\circ$	0	1	2	a
0	0	1	2	a
1	1	$\{0, 2\}$	$\{1, 2\}$	a
2	2	$\{1, 2\}$	$\{0, 1\}$	a
a	a	a	a	$\{0, 1, 2, a\}$

Hyperoperation $\circ : \mathcal{R}[3] \times \mathcal{R}[3] \mapsto \mathcal{P}(\mathcal{R}[3])$ is not continuous with the following topologies: $\Theta_1 = \{\emptyset, \mathcal{R}[3], \{1\}\}$, $\Theta_2 = \{\emptyset, \mathcal{R}[3], \{2\}\}$, $\Theta_3 = \{\emptyset, \mathcal{R}[3], \{a\}\}$, $\Theta_4 = \{\emptyset, \mathcal{R}[3], \{0, 1\}\}$, $\Theta_5 = \{\emptyset, \mathcal{R}[3], \{0, 2\}\}$, $\Theta_6 = \{\emptyset, \mathcal{R}[3], \{0, a\}\}$, $\Theta_7 = \{\emptyset, \mathcal{R}[3], \{1, 2\}\}$, $\Theta_8 = \{\emptyset, \mathcal{R}[3], \{1, a\}\}$, $\Theta_9 = \{\emptyset, \mathcal{R}[3], \{2, a\}\}$, $\Theta_{10} = \{\emptyset, \mathcal{R}[3], \{0, 1, a\}\}$, $\Theta_{11} = \{\emptyset, \mathcal{R}[3], \{0, 2, a\}\}$, $\Theta_{12} = \{\emptyset, \mathcal{R}[3], \{1, 2, a\}\}$. However, $\circ : \mathcal{R}[3] \times \mathcal{R}[3] \mapsto \mathcal{P}(\mathcal{R}[3])$ is continuous, with $\Theta_{13} = \{\emptyset, \mathcal{R}[3], \{0\}\}$, $\Theta_{14} = \{\emptyset, \mathcal{R}[3], \{0, 1, 2\}\}$. Consequently, $(\mathcal{R}[3], \Theta_{dis})$, $(\mathcal{R}[3], \Theta_{ndis})$, and $(\mathcal{R}[3], (\Theta_i)_{i=13,14})$ are soft topological polygroups. Subpolygroups of $\mathcal{R}[3]$ are $\emptyset, \mathcal{R}[3], \{0\}, \{0, 1, 2\}$. Let A be an arbitrary set, and let $a_1, a_2, a_3 \in A$. We consider a soft set $\mathbb{F}$ as

$$\mathbb{F}(x) = \begin{cases} \{0\}, & \text{if } x = a_1; \\ \{0, 1, 2\}, & \text{if } x = a_2; \\ \mathcal{R}[3], & \text{if } x = a_3; \\ \emptyset, & \text{otherwise.} \end{cases}$$

In this case, $(\mathbb{F}, A, (\Theta_i)_{i=13,14})$ is a soft topological polygroup. Let A be an arbitrary set, and let $a_1, a_2 \in A$. We define a soft set $\mathbb{F}$ as

$$\mathbb{F}(x) = \begin{cases} \{0, 1, 2\}, & \text{if } x = a_1; \\ \emptyset, & \text{if } x = a_2; \\ \{0\}, & \text{otherwise.} \end{cases}$$

Then, $(\mathbb{F}, A, \Theta_3)$ is a soft topological polygroup. We can construct many examples using this method.

Example 17. Consider $\mathcal{R}[\mathcal{R}]$:

$\circ$	0	1	2	a	b
0	0	1	2	a	b
1	1	$\{0, 2\}$	$\{1, 2\}$	a	b
2	2	$\{1, 2\}$	$\{0, 1\}$	a	b
a	a	a	a	$\{0, 1, 2, b\}$	$\{a, b\}$
b	b	b	b	$\{a, b\}$	$\{0, 1, 2, a\}$

According to Theorem 2, hyperoperation $\circ : \mathcal{R}[\mathcal{R}] \times \mathcal{R}[\mathcal{R}] \mapsto \mathcal{P}(\mathcal{R}[\mathcal{R}])$ is not continuous with the following topologies: $\Theta_1 = \{\emptyset, \mathcal{R}[\mathcal{R}], \{1\}\}$, $\Theta_2 = \{\emptyset, \mathcal{R}[\mathcal{R}], \{2\}\}$, $\Theta_3 = \{\emptyset, \mathcal{R}[\mathcal{R}], \{a\}\}$, $\Theta_4 = \{\emptyset, \mathcal{R}[\mathcal{R}], \{b\}\}$, $\Theta_5 = \{\emptyset, \mathcal{R}[\mathcal{R}], \{0, 1\}\}$, $\Theta_6 = \{\emptyset, \mathcal{R}[\mathcal{R}], \{0, 2\}\}$, $\Theta_7 = \{\emptyset, \mathcal{R}[\mathcal{R}], \{0, a\}\}$, $\Theta_8 = \{\emptyset, \mathcal{R}[\mathcal{R}], \{0, b\}\}$, $\Theta_9 = \{\emptyset, \mathcal{R}[\mathcal{R}], \{1, 2\}\}$, $\Theta_{10} = \{\emptyset, \mathcal{R}[\mathcal{R}], \{1, a\}\}$, $\Theta_{11} = \{\emptyset, \mathcal{R}[\mathcal{R}], \{1, b\}\}$, $\Theta_{12} = \{\emptyset, \mathcal{R}[\mathcal{R}], \{2, a\}\}$, $\Theta_{13} = \{\emptyset, \mathcal{R}[\mathcal{R}], \{2, b\}\}$, $\Theta_{14} = \{\emptyset, \mathcal{R}[\mathcal{R}], \{0, 1, a\}\}$, $\Theta_{15} = \{\emptyset, \mathcal{R}[\mathcal{R}], \{0, 1, b\}\}$, $\Theta_{16} = \{\emptyset, \mathcal{R}[\mathcal{R}], \{0, 1, 2\}\}$.

$\{1, 2, a\}$, $\Theta_{17} = \{\emptyset, \mathcal{R}[\mathcal{R}], \{1, 2, b\}\}$, $\Theta_{18} = \{\emptyset, \mathcal{R}[\mathcal{R}], \{2, a, b\}\}$, $\Theta_{19} = \{\emptyset, \mathcal{R}[\mathcal{R}], \{0, 1, 2, a\}\}$, $\Theta_{20} = \{\emptyset, \mathcal{R}[\mathcal{R}], \{0, 1, 2, b\}\}$, $\Theta_{21} = \{\emptyset, \mathcal{R}[\mathcal{R}], \{1, 2, a, b\}\}$. However, $\circ : \mathcal{R}[\mathcal{R}] \times \mathcal{R}[\mathcal{R}] \mapsto \mathcal{P}(\mathcal{R}[\mathcal{R}])$ is continuous with $\Theta_{22} = \{\emptyset, \mathcal{R}[\mathcal{R}], \{0\}\}$, $\Theta_{23} = \{\emptyset, \mathcal{R}[\mathcal{R}], \{0, 1, 2\}\}$. Therefore, $(\mathcal{R}[\mathcal{R}], \Theta_{dis})$, $(\mathcal{R}[\mathcal{R}], \Theta_{ndis})$, and $(\mathcal{R}[\mathcal{R}], (\Theta_i)_{i=22,23})$ are topological polygroups. Subpolygroups of $\mathcal{R}[\mathcal{R}]$ are $\emptyset, \mathcal{R}[\mathcal{R}], \{0\}$, and $\{0, 1, 2\}$. Let A be an arbitrary set, and let $a_1, a_2, a_3 \in A$. We define a soft set $\mathbb{F}$ by

$$\mathbb{F}(x) = \begin{cases} \{0\}, & \text{if } x = a_1; \\ \{0, 1, 2\}, & \text{if } x = a_2; \\ \mathcal{R}[\mathcal{R}], & \text{if } x = a_3; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then, $(\mathbb{F}, A, (\Theta_i)_{i=22,23})$ is a soft topological polygroup. Now, let A be an arbitrary set, and let $a_1, a_2 \in A$. We define a soft set $\mathbb{F}$ as

$$\mathbb{F}(x) = \begin{cases} \{0\}, & \text{if } x = a_1; \\ \emptyset, & \text{if } x = a_2; \\ \{0, 1, 2\}, & \text{otherwise..} \end{cases}$$

Then, $(\mathbb{F}, A, (\Theta_i)_{i=3,4,7,8,19,20})$ is a soft topological polygroup.

5. Conclusion

Polygroups, which are a certain subclass of hypergroups, were investigated in this study. In particular, we combined the notions of polygroups, topologies, and soft sets. Moreover, we constructed several examples of soft topological polygroups. The idea presented in this work can be applied to other algebraic hyperstructures.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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