On Mixed b -Fuzzy Topological Spaces

Wadei Faris Al-Omeri

Department of Mathematics, Al-Balqa Applied University, Salt, Jordan.

ljfis

Abstract

This paper describes the construction of a topological space from two different fuzzy topologies using the fuzzy b -q-nbd of a fuzzy point with respect to one topology and the fuzzy b -closure of a fuzzy set with respect to another topology. Additionally, some relationships are established between the two adopted topologies and the resulting mixed topology. Finally, we define countability on mixed fuzzy topological spaces and investigate the various quasi-type properties of such spaces.

Keywords: Fuzzy b -open set, Fuzzy b -cluster point, Fuzzy b -regularly open sets, Mixed topology, Fuzzy b -closure, Fuzzy b -q-nbd, Fuzzy b -closed set

1. Introduction and Preliminaries

The study of mixed topologies expanded based the work of the Polish mathematicians Alexiewicz and Semadini, but it was first invented by Finchtenholz of the Polish School of Mathematics in 1938. Topology mixing is a technique for combining two topologies on the same set to obtain a third topology. Das and Baishya [1] constructed a fuzzy topology called a mixed fuzzy topology from two given topologies on a set $\mathcal{X}$ based on the closure of the neighborhoods of one topology with respect to the other topology and then studied various properties of the resulting topology.

Different properties of fuzzy topological spaces have been investigated by Arya and Singal [2, 3], Chang [4], Das and Baishya [1], Ganster et al. [5], Ganguly and Singha [6], Ghanim et al. [7], Tripathy and Debnath [8], etc. Recently, mixed fuzzy topological spaces have been investigated from different perspectives by Das and Baishya [1], Tripathy and Ray [9, 10] etc.

Azad [11] defined the concept of fuzzy regular open and fuzzy regular closed sets. Benchalli and Karnel [12] defined the concepts of fuzzy b -open sets and fuzzy b -continuous mappings in fuzzy topological spaces. In this paper, we introduce and study the concept of mixed b -fuzzy sets on mixed fuzzy topological spaces. We investigate this concept in the context of b -q-neighborhoods, b -q-coincidence, fuzzy b -closures, and fuzzy b -interiors. Additionally, some relations are established between the two main topologies considered and their corresponding mixed fuzzy topological spaces. Most of the concepts, notations, and definitions used in this paper are standard, but for the sake of completeness, we recall some basic definitions and results below.

Definition 1.1 ([4]). A fuzzy point in $\mathcal{X}$ is a fuzzy set in $\mathcal{X}$ and is zero everywhere, except at one point (e.g., x), where it takes value such as r with $r \in (0, 1)$ (i.e., $(0 < r < 1)$). We denote this set as x_r , where the point x is its support and r is its value.

Received: Dec. 21, 2018

Revised : Apr. 1, 2020

Accepted: Sep. 9, 2020

Correspondence to: Wadei Faris Al-Omeri
 (wadeimoon1@hotmail.com, wadeialomeri@bau.edu.jo)

©The Korean Institute of Intelligent Systems

©This is an Open Access article distributed under the terms of the Creative Commons Attribution Non-Commercial License (<http://creativecommons.org/licenses/by-nc/3.0/>) which permits unrestricted non-commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

Definition 1.2 ([4]). A fuzzy set A in $\mathcal{X}$ is called q-coincident with a fuzzy set B in $\mathcal{X}$, denoted as AqB , if and only if $A(x) + B(x) > 1$ for some $x \in \mathcal{X}$. It is clear that if $x_r qA$, then $r + A(x) \leq 1$ for every $x \in \mathcal{X}$ and if AqB , then $A(x) + B(x) \leq 1$ for every $x \in \mathcal{X}$.

Let A be a fuzzy subset of a space $\mathcal{X}$. The fuzzy closure of A and fuzzy interior of A are denoted as $Cl(A)$ and $Int(A)$, respectively, and are defined by $Cl(r) = \cap\{u : u \text{ is a closed fuzzy set in } \mathcal{X} \text{ and } A \subseteq u\}$ and $Int(r) = \cap\{v : v \text{ is an open fuzzy set in } \mathcal{X} \text{ and } v \subseteq r\}$, respectively. Clearly, $Cl(r)$ ($Int(r)$) is the smallest (largest) closed (open) fuzzy set in X containing (contained in) r . If there is more than one topologies on X , then the closure and interior of r with respect to a fuzzy topology τ on X are denoted as $\tau - Cl(r)$ and $\tau - Int(r)$, respectively. A fuzzy subset r of a space X is called fuzzy-pre-open [13] if $r \leq Int(Cl(r))$. The complement of a fuzzy-pre-open set is called a fuzzy-pre-closed set.

Definition 1.3. A fuzzy singleton x_r in $\mathcal{X}$ is said to be quasi-coincident (q-coincident) with a fuzzy set A in $\mathcal{X}$, denoted as $x_r qA$, if and only if $r + A(x) > 1$.

Definition 1.4. A fuzzy set A in a fuzzy time series (FTS) $(\mathcal{X}, \mathcal{T})$ is called a quasi-neighborhood (q-nbd) of x_r if and only if $A_1 \in \mathcal{T}$ such that $A_1 \subseteq A$ and $x_r qA_1$. The family of all q-neighborhoods of x_r is called the system of q-neighborhoods of x_r . The intersection of two quasi-neighborhoods of x_r is also a quasi-neighborhood. Let $(\mathcal{X}, \mathcal{T}_1)$ and $(\mathcal{X}, \mathcal{T}_2)$ be two fuzzy topological spaces and let $\mathcal{T}_1(\mathcal{T}_2) = \{A \in I^{\mathcal{X}} : \text{for every fuzzy set } B \text{ in } \mathcal{X} \text{ with } AqB, \text{ there exists a } \mathcal{T}_2\text{-q-nbd } A_\alpha \text{ such that } A_\alpha qB \text{ and we have a } \mathcal{T}_1\text{-closure } A_\alpha \subseteq B. \text{ Then, } \mathcal{T}_1(\mathcal{T}_2) \text{ is a fuzzy topology on } \mathcal{X} \text{ called a mixed fuzzy topology and the space } (\mathcal{X}, \mathcal{T}_1(\mathcal{T}_2)) \text{ is called a mixed fuzzy topological space.}$

Definition 1.5. A fuzzy set A in a fuzzy topological space $\mathcal{X}$ is called fuzzy b -open [14] if $A \leq Int(Cl(A)) \vee Cl(Int(A))$. The complement of a fuzzy b -open set is called a fuzzy b -closed set (i.e., $Int(Cl(A)) \wedge Cl(Int(A)) \geq A$).

Definition 1.6 ([12]). A fuzzy set A in a fuzzy topological space $\mathcal{X}$ is called a fuzzy b -q-nbd of a fuzzy point x_r in $\mathcal{X}$ if there is a fuzzy b -open set V in X such that $x_r qV \leq A$. Additionally, if A is fuzzy b -open, then A will be referred to as a fuzzy b -open q-nbd of x_r .

Lemma 1.7[14]. In a fuzzy topological space $(\mathcal{X}, \mathcal{T})$. Every fuzzy pre-open set is also fuzzy b -open.

Definition 1.8 ([11, 14]). Let A be a fuzzy set in an FTS $\mathcal{X}$. Then, $bCl(A) = \wedge\{u \geq A : u \text{ is a } fb\text{-closed set of } \mathcal{X}\}$.

Additionally, because $bInt(A) = \vee\{v \leq A, v \text{ is an } fb\text{-open set of } \mathcal{X}\}$.

Theorem 1.9 ([14]). An FTS $\mathcal{X}$ A is fb -open (fb -closed) if and only if $A = fbCl(S)$ ($fbInt(S)$).

Remark 1.10 ([14]). A fuzzy set A in a fuzzy topological space $\mathcal{X}$ is fuzzy e -closed if and only if $A = bCl(A)$.

2. Main Results

In this section, we discuss the construction of a mixed fuzzy topology with help from the fuzzy b -q-nbd of a fuzzy point.

Definition 2.1. A fuzzy point x_r in a fuzzy topological space $\mathcal{X}$ is called a fuzzy b -cluster point of a fuzzy set A in $\mathcal{X}$ if every fuzzy b -q-nbd of x_r is q-coincident with A . The union of all fuzzy b -cluster points in A is called the fuzzy b -closure of A and is denoted as $bCl(A)$.

Theorem 2.2. Let $(\mathcal{X}, \mathcal{T}_1(\mathcal{T}_2))$ be a mixed b -fuzzy topological space. If $U_{\alpha\alpha} = \{U \in I^{\mathcal{X}} : \text{for every fuzzy point } x_\alpha, \text{ there exists a fuzzy } \mathcal{T}_2 - b\text{-set } V \text{ in } \mathcal{X} \text{ such that } x_\alpha qV \text{ and } \mathcal{T}_1 - bCl(V) \leq U\}$, then there is a fuzzy topology $\mathcal{T}_1(\mathcal{T}_2)$ for which $U_{\alpha\alpha}$ is a fuzzy b -q-nbd system of x_α under the condition that the intersection of two b -fuzzy sets is also b -fuzzy.

Proof. (i) Let x_α be a fuzzy point and $U \in U_{\alpha\alpha}$. We must demonstrate that $x_\alpha qU$. Because $U \in U_{\alpha\alpha}$, there is a fuzzy $\mathcal{T}_2$ - b -set V in X such that $x_\alpha qV$ and $\mathcal{T}_1 - bCl(V) \leq U$. Now, $x_\alpha qV$ implies $\alpha + V(x) > 1$. Additionally, $V \leq 1 - bCl(V) \leq U$. Therefore, $V(x) \leq U(x)$. Finally, $\alpha + U(x) \geq \alpha + V(x) > 1$, which implies $\alpha + U(x) > 1$, meaning we have $x_\alpha qU$.

(ii) Let $U, V \in U_{\alpha\alpha}$. Then, there are fuzzy $\mathcal{T}_2$ - b -sets U_1 and V_1 such that $x_\alpha qU_1$ with $\mathcal{T}_1 - bCl(U_1) \leq U$ and $x_\alpha qV_1$ with $\mathcal{T}_1 - bCl(V_1) \leq V$, respectively. Now, $x_\alpha qU_1$ and $x_\alpha qV_1$

$$\Rightarrow \alpha + U_1(x) > 1 \text{ and } \alpha + V_1(x) > 1$$

$$\Rightarrow \alpha + \min\{U_1(x), V_1(x)\} > 1 \Rightarrow \alpha + (U_1 \cap V_1)(x) > 1$$

$$\Rightarrow x_\alpha q(U_1 \cap V_1).$$

Because U_1 and V_1 are fuzzy $\mathcal{T}_2$ - b -sets, according to the given condition, $(U_1 \cap V_1)$ is a fuzzy $\mathcal{T}_2$ - b -set. Additionally, $\mathcal{T}_1 - bCl(U_1 \cap V_1) \leq \mathcal{T}_1 - bCl(U_1) \cap \mathcal{T}_1 - bCl(V_1) \leq U \cap V$. Therefore, $U \cap V \in U_{\alpha\alpha}$.

(iii) Let U and V be two fuzzy sets such that $U \in U_{\alpha\alpha}$ and $U \leq V$. We must demonstrate that $V \in U_{\alpha\alpha}$. Because $U \in U_{\alpha\alpha}$, there is a fuzzy $\mathcal{T}_2$ - b -set U_1 such that $x_\alpha qU_1$ and $\mathcal{T}_1 - bCl(U_1) \leq U \leq V$. Therefore, $V \in U_{\alpha\alpha}$.

Therefore, $\mathcal{T}_1(\mathcal{T}_2)$ is a fuzzy topology on $\mathcal{X}$ for which $U_{\alpha\alpha}$ is a fuzzy b - q -nbd system of x_α . $\square$

Definition 2.3. Let $(\mathcal{X}, \mathcal{T}_1)$ and $(\mathcal{X}, \mathcal{T}_2)$ be two fuzzy topological spaces. We define $\mathcal{T}_1(\mathcal{T}_2) = \{A \in I^\mathcal{X} \mid \text{for every } x_\alpha q A \text{ and there is a fuzzy } \mathcal{T}_2\text{-}b\text{-}q\text{-nbd } A_\alpha \text{ of } x_\alpha \text{ such that } \mathcal{T}_1\text{-}bCl(A_\alpha) \subseteq A\}$. Then, $\mathcal{T}_1(\mathcal{T}_2)$ is a fuzzy topology on $\mathcal{X}$. This fuzzy topology is called a mixed b -fuzzy topology and $(\mathcal{X}, \mathcal{T}_1(\mathcal{T}_2))$ is called a mixed b -fuzzy topological space.

Theorem 2.4. The intersection of two fuzzy b - q -nbds of a fuzzy point α is also a fuzzy b - q -nbd of α in the fuzzy topological spaces satisfying the condition that the intersection of two fuzzy b -open sets is fuzzy b -open.

Proof. Let $(\mathcal{X}, \mathcal{T}_1)$ be a fuzzy topological space satisfying the condition that the intersection of two fuzzy b -open sets is fuzzy b -open. Let A and B be two fuzzy b - q -nbds of a fuzzy point α . Therefore, there are fuzzy b -open sets U_1 and U_2 such that $\alpha q U_1 \leq A$ and $\alpha q U_2 \leq B$. Therefore, $\alpha + u_1(x) > 1$ and $\alpha + u_2(x) > 1$, which implies $\alpha + \min\{u_1(x), u_2(x)\} > 1$. Now, $\alpha + (U_1 \cap U_2)(x) > 1$ and $\alpha q (U_1 \cap U_2)$. Because $U_1 \leq A$ and $U_2 \leq B$, we have $U_1 \cap U_2 \leq A \cap B$. Additionally, $U_1 \cap U_2$ is fuzzy b -open because the intersection of two fuzzy b -open sets is also fuzzy b -open based on our assumptions. By combining this result with (1) and (2), we can conclude that $U_1 \cap U_2$ is a fuzzy b - q -nbd of α , which completes the proof. $\square$

Theorem 2.5. Let $(\mathcal{X}, \mathcal{T}_1)$ and $(\mathcal{X}, \mathcal{T}_2)$ be two fuzzy topological spaces. Consider the collection of fuzzy sets $\mathcal{T}_1(\mathcal{T}_2) = \{A \in I^\mathcal{X} \mid \text{For any fuzzy set } B \text{ in } \mathcal{X} \text{ with } A q B, \text{ there is a } \mathcal{T}_2\text{-}b\text{-}q\text{-nbd set } A_1 \text{ such that } A_1 q B \text{ and } \mathcal{T}_1\text{-}bCl(A_1) \subseteq A\}$. Then, this family of fuzzy sets forms a topology on $\mathcal{X}$ and this topology is called a mixed fuzzy topology on $\mathcal{X}$.

Proof. ($\mathcal{T}_1$) Here, $1_\mathcal{X} \in \mathcal{T}_1(\mathcal{T}_2)$. Because $\alpha + 1_\mathcal{X} > 1$, for any fuzzy point x_α , $x_\alpha q 1_\mathcal{X}$ and for any fuzzy $\mathcal{T}_2\text{-}b\text{-}q\text{-nbd } A_\alpha = 1_\mathcal{X}$ of x_α , $\mathcal{T}_1\text{-}bCl(A_\alpha) = \mathcal{T}_1\text{-}bCl(1_\mathcal{X}) = 1_\mathcal{X}$. Additionally, $0_\mathcal{X} \in \mathcal{T}_1(\mathcal{T}_2)$ because $0_\mathcal{X} \in \mathcal{T}_1(\mathcal{T}_2)$ is not q -coincident with any fuzzy point for $\alpha + 0_\mathcal{X} \not> 1$, meaning there is no doubt of the violation of the condition of being a member of $\mathcal{T}_1(\mathcal{T}_2)$. Therefore, $1_\mathcal{X}, 0_\mathcal{X} \in \mathcal{T}_1(\mathcal{T}_2)$.

($\mathcal{T}_2$) Let $\{A_\lambda : \lambda \in \Delta\}$ be a collection of b -fuzzy sets in $\mathcal{T}_1(\mathcal{T}_2)$. We must demonstrate that $\bigcup_\lambda A_\lambda \in \mathcal{T}_1(\mathcal{T}_2)$. Let $\{A_\lambda\}$ be a collection of members of $\mathcal{T}_1(\mathcal{T}_2)$. We must also demonstrate that for every $x_\alpha q \bigcup_\lambda A_\lambda$, there is a fuzzy $\mathcal{T}_2\text{-}b\text{-}q\text{-nbd } A_\alpha$ of x_α such that $\mathcal{T}_1\text{-}bCl(A_\alpha) \subseteq \bigcup_\lambda A_\lambda$. Consider a fuzzy point x_α . Now, $x_\alpha q \bigcup_\lambda A_\lambda$ implies $\alpha + \bigcup_\lambda A_\lambda > 1$. This

implies $\alpha + \sup_\lambda A_\lambda > 1$, which in turn implies $\sup_\lambda A_\lambda > 1 - \alpha$. Therefore, there is a λ_0 such that $\alpha + A_{\lambda_0}(x) > 1$. This implies $A_{\lambda_0}(x) > 1 - \alpha$. In other words, $x_\alpha q A_{\lambda_0}$. We know that $A_{\lambda_0} \in \mathcal{T}_1(\mathcal{T}_2)$. Therefore, there is a fuzzy $\mathcal{T}_2\text{-}b\text{-}q\text{-nbd } A_{\gamma_0}$ of x_α such that $\mathcal{T}_1\text{-}bCl(A_{\gamma_0}) \subseteq A_{\lambda_0} \subseteq \bigcup_\lambda A_\lambda$. Therefore, $\bigcup_\lambda A_\lambda \in \mathcal{T}_1(\mathcal{T}_2)$.

($\mathcal{T}_3$) We must prove that $(A_1 \cap A_2) \in \mathcal{T}_1(\mathcal{T}_2)$ where $A_1, A_2 \in \mathcal{T}_1(\mathcal{T}_2)$. In other words, we must prove that for every $x_\beta q (A_1 \cap A_2)$, there is a fuzzy $\mathcal{T}_2\text{-}b\text{-}q\text{-nbd } A_\beta$ of x_β such that $\mathcal{T}_1\text{-}bCl(A_\beta) \subseteq (A_1 \cap A_2)$. Let $x_\beta q (A_1 \cap A_2)$. This implies $\beta + (A_1 \cap A_2)(x) > 1$, which implies $\beta + \min\{A_1(x), A_2(x)\} > 1$. Therefore, $\beta + A_1(x) > 1$ and $\beta + A_2(x) > 1$, meaning $x_\beta q A_1$ and $x_\beta q A_2$. Now $A_1 \in \mathcal{T}_1(\mathcal{T}_2)$. This implies that there is a fuzzy $\mathcal{T}_2\text{-}b\text{-}q\text{-nbd } A_{\beta_1}$ of x_β such that $\mathcal{T}_1\text{-}bCl(A_{\beta_1}) \subseteq A_1$. Additionally, $A_2 \in \mathcal{T}_1(\mathcal{T}_2)$, which implies that there is a fuzzy $\mathcal{T}_2\text{-}b\text{-}q\text{-nbd } A_{\beta_2}$ of x_β such that $\mathcal{T}_1\text{-}bCl(A_{\beta_2}) \subseteq A_2$. Because A_{β_1} and A_{β_2} are fuzzy $\mathcal{T}_2\text{-}b\text{-}q\text{-nbds}$ of x_β , we know $A_{\beta_1} \cap A_{\beta_2}$ is a fuzzy $\mathcal{T}_2\text{-}b\text{-}q\text{-nbd}$ of x_β according to Lemma 2.4. Based on this intersection, we get $(\mathcal{T}_1\text{-}bCl(A_{\beta_1})) \cap (\mathcal{T}_1\text{-}bCl(A_{\beta_2})) \subseteq (A_1 \cap A_2)$. Finally, we must prove that $\mathcal{T}_1\text{-}bCl(A_{\beta_1} \cap A_{\beta_2}) \subseteq (\mathcal{T}_1\text{-}bCl(A_{\beta_1})) \cap (\mathcal{T}_1\text{-}bCl(A_{\beta_2}))$. Let $x_\delta \in bCl(A_{\beta_1} \cap A_{\beta_2})$. This implies that $A_{\gamma_1} \cap A_{\gamma_2}$ is a b -neighborhood of x_γ , which implies that every fuzzy b - q -nbd U of x_γ is q -coincident with $(A_{\gamma_1} \cap A_{\gamma_2})$. Therefore, $U q (A_{\gamma_1} \cap A_{\gamma_2})$, which implies $U(x) + (A_{\beta_1} \cap A_{\beta_2})(x) > 1$, meaning $U(x) + \min(A_{\beta_1}(x) \cap A_{\beta_2}(x)) > 1$. Therefore, $U(x) + A_{\beta_1}(x) > 1$ and $U(x) + A_{\beta_2}(x) > 1$. This implies that every fuzzy b - q -nbd U of x_δ is q -coincident with A_{γ_1} and A_{γ_2} . Therefore, x_δ is a fuzzy point in A_{β_1} and A_{β_2} . Therefore, $x_\gamma \in bCl(A_{\beta_1})$ and $x_\delta \in bCl(A_{\beta_2})$. So $x_\delta \in bCl(A_{\beta_1}) \cap bCl(A_{\beta_2})$, meaning $bCl(A_{\beta_1} \cap A_{\beta_2}) \subseteq bCl(A_{\beta_1}) \cap bCl(A_{\beta_2}) \subseteq A_1 \cap A_2$ according to (5). Therefore, $A_1 \cap A_2 \in \mathcal{T}_1(\mathcal{T}_2)$, where $A_1, A_2 \in \mathcal{T}_1(\mathcal{T}_2)$, meaning $\mathcal{T}_1(\mathcal{T}_2)$ is a fuzzy topology on $\mathcal{X}$. $\square$

Lemma 2.6. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two fuzzy topologies on a set $\mathcal{X}$. If every fuzzy $\mathcal{T}_1\text{-}b\text{-}q\text{-nbd}$ of x_p is a fuzzy $\mathcal{T}_2\text{-}b\text{-}q\text{-nbd}$ of x_p , then $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$.

Proof. Let A be a fuzzy $\mathcal{T}_1\text{-}b\text{-}q\text{-nbd}$ of x_p . This indicates that there is a fuzzy $\mathcal{T}_1\text{-}b\text{-open}$ set M in $\mathcal{X}$ such that $x_p q M$ and $M \leq A$. Now, A is fuzzy $\mathcal{T}_1$ -open, which implies that A is fuzzy $\mathcal{T}_1$ -pre-open, which also implies that A is fuzzy $\mathcal{T}_1$ - b -open according to Lemma 1.7. Therefore, we can assume $M = A$. Let A be a fuzzy $\mathcal{T}_1\text{-}b\text{-}q\text{-nbd}$ of x_p that is q -coincident with A . According to our hypothesis, A is a fuzzy $\mathcal{T}_2\text{-}b\text{-}q\text{-nbd}$ of x_p . Therefore, there is a fuzzy $\mathcal{T}_2\text{-}b\text{-open}$ set M_1 such that $x_p q M_1$ and $M_1 \leq A$. We must prove that A is fuzzy

$\mathcal{T}_1$ -open. To this end, we prove that $A(x) = \sup M_1(x)$ with the supremum being taken over all $M_1 \leq A$. Now, based on the fact that $A(x) + 1 - A(x) + \varepsilon > 1$, it follows that $x_{1+\varepsilon-A(x)}qA$. Therefore, $x_{1+\varepsilon-A(x)}qM_1$ for some $M_1 \in \mathcal{T}_2$ with $M_1(x) \leq A(x)$. Consequently, $1 + \varepsilon - A(x) + M_1(x) > 1$ or $M_1(x) > A(x) - \varepsilon$. Therefore, $A(x) = \sup M_1(x)$ and each M_1 is fuzzy $\mathcal{T}_2$ -open. Therefore, each A is fuzzy $\mathcal{T}_2$ -open and $\mathcal{T}_1 \leq \mathcal{T}_2$, which completes the proof. $\square$

Lemma 2.7. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two fuzzy topologies on a set $\mathcal{X}$. Then, the mixed b -fuzzy topology $\mathcal{T}_1(\mathcal{T}_2)$ is coarser than $\mathcal{T}_2$, which is denoted as $\mathcal{T}_1(\mathcal{T}_2) \leq \mathcal{T}_2$.

Proof. Let A be a fuzzy $\mathcal{T}_1(\mathcal{T}_2)$ - b - q -nbd of x_p . We must prove that A is a fuzzy $\mathcal{T}_2$ - b - q -nbd of x_p . A is a fuzzy $\mathcal{T}_1(\mathcal{T}_2)$ - b - q -nbd of x_p , which implies that there is a fuzzy $\mathcal{T}_1(\mathcal{T}_2)$ - b -open set M in $\mathcal{X}$ such that x_pqM and $M \leq A$. In other words, for every x_pqM , there is a fuzzy $\mathcal{T}_2$ - b - q -nbd M_1 of x_p such that $\mathcal{T}_1 - bCl(M_1) \leq M$. This implies that there is a fuzzy $\mathcal{T}_2$ - b -open set M_1 such that $x_pqM_2 \leq M_1 \leq \mathcal{T}_1 - bCl(M_1) \leq M \leq A$. Therefore, A is a fuzzy $\mathcal{T}_1(\mathcal{T}_2)$ - b - q -nbd of x_p , meaning $\mathcal{T}_1(\mathcal{T}_2) \leq \mathcal{T}_2$ according to Lemma 2.6, which completes the proof. $\square$

Definition 2.8. A fuzzy topological space $(\mathcal{X}, \mathcal{T})$ is said to be fuzzy b -regular if for each fuzzy point x_α and each fuzzy b -closed set W with $x_\alpha q(1 - W)$, there is a fuzzy open set A and fuzzy b -open set B such that $x_\alpha qA$, $W \leq B$, and A is not q -coincident with B .

Application 2.9. Theorem 2.7 can be applied to represent the open subsets of the mixed topology $\mathcal{T}_1(\mathcal{T}_2)$. For example, if $X = \{a, b\}$, $\mathcal{T}_1 = \{1, 0, (a, .4)\}$, then $\mathcal{T}_2 = \{1, 0, (b, .3)\}$ and according to Theorem 2.7, $\mathcal{T}_1(\mathcal{T}_2) \leq \mathcal{T}_2$. To find the exact members of $\mathcal{T}_1(\mathcal{T}_2)$, it is sufficient to determine if b belongs to $\mathcal{T}_1(\mathcal{T}_2)$. In fact, $(b, .3) \notin \mathcal{T}_1(\mathcal{T}_2)$. Therefore, $\mathcal{T}_1(\mathcal{T}_2) = \{1, 0\} \neq \mathcal{T}_2$. This also implies that $\mathcal{T}_1(\mathcal{T}_2)$ is strictly coarser than $\mathcal{T}_2$. We now present another example where $\mathcal{T}_1(\mathcal{T}_2) = \mathcal{T}_2$.

Example 2.10. Let us consider a non-empty set $X = \{a, b\}$ and consider the following fuzzy sets in X : $C = \{(a, .7), (b, .3)\}$, and $D = \{(a, .3), (b, .7)\}$. Then, the collection of fuzzy sets $\mathcal{T}_1 = \{0', 1', D\}$, and $\mathcal{T}_2 = \{0', 1', C\}$ represent two fuzzy topologies on X . Now, we construct a mixed fuzzy topology on X from the two fuzzy topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. Because the mixed fuzzy topology is coarser than $\mathcal{T}_2$, we only need to verify that the fuzzy set A is in $\mathcal{T}_1(\mathcal{T}_2)$.

Let us consider a fuzzy set G in X such that CqG . Now, the only $\mathcal{T}_2$ -open sets are $1'$ and C such that CqG and $1'qG$.

This means that the $\mathcal{T}_1$ -closure of $C = \wedge\{Q : Q \text{ is } \mathcal{T}_1 \text{ - closed and } A \subseteq Q\} = 1' \wedge C = C \subseteq C$. Therefore, $A \subseteq \mathcal{T}_1(\mathcal{T}_2)$ and $\mathcal{T}_1(\mathcal{T}_2) = \{0', 1', C\}$. In this mixed fuzzy topology, the fuzzy regular open sets are $0', 1'$, but C is not fuzzy regular open because $\mathcal{T}_1 - Int(\mathcal{T}_2 - Cl(C)) = 1'$. Now, we demonstrate that C is fuzzy b -open. It is sufficient to prove that $\overline{C} = D$ is fuzzy b -closed. Now, $\mathcal{T}_1 - Cl(\mathcal{T}_2 - Int(C)) = A$. Therefore, D is fuzzy b -closed and C is fuzzy b -open. Therefore, we can conclude that a fuzzy b -open set need not be fuzzy regular open.

We now establish a relationship between $\mathcal{T}_1$ and $\mathcal{T}_1(\mathcal{T}_2)$ for some specific cases.

Proposition 2.11. If $\mathcal{T}_1$ is fuzzy b -regular and $\mathcal{T}_1 \leq \mathcal{T}_2$, then $\mathcal{T}_1 \leq \mathcal{T}_1(\mathcal{T}_2)$.

Proof. Let A be a fuzzy $\mathcal{T}_1$ - b - q -nbd of x_p . This implies that there is a fuzzy $\mathcal{T}_1$ - b -open set M in X such that x_pqM and $M \leq A$. We must prove that A is a fuzzy $\mathcal{T}_1(\mathcal{T}_2)$ - b - q -nbd of x_p . Because $\mathcal{T}_1$ is fuzzy b -regular, for each x_p and for each fuzzy b -open set M that is q -coincident with x_p , there is a fuzzy $\mathcal{T}_1$ open set M_1 in X such that $x_pqM_1 \leq bCl(M_1) \leq M$. Now, a fuzzy set being fuzzy open implies that the set is fuzzy pre-open, which implies that the set is fuzzy b -open according to Lemma 1.7. Therefore, M_1 is fuzzy b -open. Again, $\mathcal{T}_1 \leq \mathcal{T}_2$ implies that M_1 is $\mathcal{T}_2$ fuzzy open. Therefore, M_1 is $\mathcal{T}_1$ -fuzzy open, x_pqM_1 , and $bCl(M_1) \leq M \leq A$. Therefore, M_1 is $\mathcal{T}_1(\mathcal{T}_2)$ -fuzzy open, meaning A is a fuzzy $\mathcal{T}_1(\mathcal{T}_2)$ - b - q -nbd of x_p and $\mathcal{T}_1 \leq \mathcal{T}_1(\mathcal{T}_2)$ according to Lemma 2.6, which completes the proof. $\square$

3. Conclusion

We introduced mixed b -fuzzy topological spaces and a fuzzy completely weak b -irresolute function over an initial universe with a fixed set of parameters. Many results have been established to demonstrate how topological structures are preserved by this b -irresolute function. We also provided examples in which such properties fail to be preserved. In this paper, we only focused on a few specific ideas. In the future, it will be necessary to conduct more theoretical research to establish a general framework for practical applications.

References

- [1] N. R. Das and P. C. Baishya, "Mixed fuzzy topological spaces," *Journal of Fuzzy Mathematics*, vol. 3, no. 4, pp. 777-784, 1995.

- [2] S. P. Arya and N. Jain, "Fuzzy subweakly continuous functions," *Mathematics Student-India*, vol. 70, no. 1-4, pp. 231-240, 2001.
- [3] S. P. Arya and N. Jain, "Fuzzy subweakly a-continuous functions," *Mathematics Student-India*, vol. 70, no. 1-4, pp. 241-246, 2001.
- [4] C. L. Chang, "Fuzzy topological spaces," *Journal of Mathematical Analysis and Applications*, vol. 24, no. 1, pp. 182-190, 1968.
- [5] M. Ganster, D. N. Georgiou, S. Jafari, and S. P. Moshokoa, "On some applications of fuzzy points," *Applied General Topology*, vol. 6, no. 2, pp. 119-133, 2005. <https://doi.org/10.4995/agt.2005.1951>
- [6] S. Ganguly and D. Singha, "Mixed topology for a bi-topological spaces," *Bulletin of the Calcutta Mathematical Society*, vol. 76, pp. 304-314, 1984.
- [7] M. H. Ghanim, E. E. Kerre, and A. S. Mashhour, "Separation axioms, subspaces and sums in fuzzy topology," *Journal of Mathematical Analysis and Applications*, vol. 102, no. 1, pp. 189-202, 1984.
- [8] B. C. Tripathy and A. Baruah, "New type of difference sequence spaces of fuzzy real numbers," *Mathematical Modelling and Analysis*, vol. 14, no. 3, pp. 391-397, 2009. <https://doi.org/10.3846/1392-6292.2009.14.391-397>
- [9] B. C. Tripathy and A. Baruah, "Nörlund and Riesz mean of sequences of fuzzy real numbers," *Applied Mathematics Letters*, vol. 23, no. 5, pp. 651-655, 2010. <https://doi.org/10.1016/j.aml.2010.02.006>
- [10] B. C. Tripathy and G. C. Ray, "Weakly continuous functions on mixed fuzzy topological spaces," *Acta Scientiarum Technology*, vol. 36, no. 2, pp. 331-335, 2014. <https://doi.org/10.4025/actascitechnol.v36i2.16241>
- [11] K. K. Azad, "On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity," *Journal of Mathematical Analysis and Applications*, vol. 82, no. 1, pp. 14-32, 1981. [https://doi.org/10.1016/0022-247X\(81\)90222-5](https://doi.org/10.1016/0022-247X(81)90222-5)
- [12] S. S. Benchalli and J. J. Karna, "On fuzzy b-neighbourhoods and fuzzy b-mappings in fuzzy topological spaces," *Journal of Computer and Mathematical Sciences*, vol. 1, no. 6, pp. 696-701, 2010.
- [13] A. S. Bin Shahna, "On fuzzy strong semicontinuity and fuzzy precontinuity," *Fuzzy Sets and Systems*, vol. 44, no. 2, pp. 303-308, 1991. [https://doi.org/10.1016/0165-0114\(91\)90013-G](https://doi.org/10.1016/0165-0114(91)90013-G)
- [14] S. S. Benchalli and J. J. Karna, "On fuzzy b-open sets in fuzzy topological spaces," *Journal of Computer and Mathematical Sciences*, vol. 1, no. 2, pp. 103-273, 2010.
- [15] B. C. Tripathy and S. Debnath, " γ -open sets and γ -continuous mappings in fuzzy bitopological spaces," *Journal of Intelligent & Fuzzy Systems*, vol. 24, no. 3, pp. 631-635, 2013. <https://doi.org/10.3233/IFS-2012-0582>
- [16] B. C. Tripathy and G. C. Ray, "On mixed fuzzy topological spaces and countability," *Soft Computing*, vol. 16, no. 10, pp. 1691-1695, 2012. <https://doi.org/10.1007/s00500-012-0853-1>

Wadei Faris Al-Omeri is a Mathematics Assistant Professor at the Al-Balqa Applied University recommend for the Award 2020 Mmathematics for Young Investigators. Dr. Wadei has proven to be an excellent researcher and a wonderful

asset to the academic profession. Author obtained his highest degree from the most prestigious university and work in the best organization for over a decade. He has published original articles in the finest journal in the area of his study. His research interests include (but not limited to) General Topology, Fuzzy Topology, Neutrosophic Tpology. He graduated with a PhD in Mathematics specializing in the area of research of Topology. He has excellent output from his PhD work and currently is still having very good momentum in producing new results. In fact, many of his work has good citation statistics. He has done many research collaborations with fellow colleagues and also acts as mentors' to junior ones. Apart from having a strong research interest, Dr. Wadei is also very passionate about teaching. He is much loved by students because of his never ending effort to make them at ease with the many difficult concepts of mathematics. On the whole, he works hard, loves students and the research environment and is very comfortable with academia. His warm personality is also very much appreciated by many. E-mail: wadeimoon1@hotmail.com, wadeialomeri@bau.edu.jo