On Separation Axioms in Fuzzifying Bitopological Spaces

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Abstract

In present paper, we introduce and study the concepts of $T_0^{(i,j)}$ -, $T_1^{(i,j)}$ -, $T_2^{(i,j)}$ (pairwise Hausdorff)-, $T_R^{(i,j)}$ (pairwise regularity)-, $T_N^{(i,j)}$ (pairwise normality)-, $R_0^{(i,j)}$ -, $R_1^{(i,j)}$ -, $T_3^{(i,j)}$ - and $T_4^{(i,j)}$ - separation axioms in fuzzifying bitopological spaces and study some relations between them. Also we investigate the image of these kinds of fuzzifying bitopological spaces under some types of fuzzy mappings.

Keywords: Fuzzifying topology, Fuzzifying bitopological space, Separation axioms

1. Introduction

In 1963, Kelley [1] introduced the notions of bitopological spaces. Such spaces equipped with its two (arbitrary) topologies. In 1991-1993, Ying [2-4] introduced the concept of the fuzzifying topology with the semantic method of continuous valued logic. On the framework of fuzzifying topology, Shen [5] introduced and studied the notions of separation axioms. In 2003, Zhang and Liu [6] studied the concepts of fuzzy $\theta_{(i,j)}$ -closed, $\theta_{(i,j)}$ -open sets in fuzzifying bitopological spaces. In 2007, Kiliciman and Salleh [7] introduced two concepts of pairwise Landelöf bitopological spaces and studied the properties of them.

The contents of this paper are arranged as follows. In Section 3, we introduce $T_0^{(i,j)}$ -, $T_1^{(i,j)}$ -, $T_2^{(i,j)}$ (pairwise Hausdorff)-, $T_R^{(i,j)}$ (regularity)-, $T_N^{(i,j)}$ (pairwise normality)-, $R_0^{(i,j)}$ - and $R_1^{(i,j)}$ - separation axioms in fuzzifying bitopological spaces. We define $T_3^{(i,j)}$, $T_4^{(i,j)}$ and new weaker form of the pairwise normality axioms in fuzzifying bitopological spaces. In Section 4, we study some important relations between that separation axioms in fuzzifying bitopological spaces. Finally in Section 5, we investigate the image of these kinds of fuzzifying bitopological spaces under some types of fuzzy mappings.

2. Preliminaries

Firstly, we display the fuzzy logical and corresponding set-theoretical notations used in this paper. For formula φ , the symbol $[\varphi]$ means the truth of φ , where the set of truth values is the unit interval $[0, 1]$. A formula φ is valid, we write $\models \varphi$ if and only if $[\varphi] = 1$ for every interpretation.

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(1) $[\alpha] := \alpha$ ($\alpha \in [0, 1]$); $[\alpha \wedge \beta] = \min([\alpha], [\beta])$; $[\alpha \rightarrow \beta] = \min(1, 1 - [\alpha] + [\beta])$, $[\forall x \alpha(x)] = \inf_{x \in X} [\alpha(x)]$,

where X is the universe of discourse.

(2) If $\tilde{A} \in \mathfrak{S}(X)$, where $\mathfrak{S}(X)$ is the family of fuzzy sets of X , then $[x \in \tilde{A}] := \tilde{A}(x)$.

(3) If X is the universe of discourse, then $[\forall x \alpha(x)] = \inf_{x \in X} [\alpha(x)]$.

In addition, the following derived formulae are given:

(1) $[\neg \alpha] = 1 - [\alpha]$.

(2) $[\alpha \vee \beta] = \max([\alpha], [\beta])$.

(3) $[\alpha \leftrightarrow \beta] := [(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)]$.

(4) $[\alpha \dot{\wedge} \beta] = \max(0, [\alpha] + [\beta] - 1)$.

(5) $[\alpha \dot{\vee} \beta] = \min(1, [\alpha] + [\beta])$.

(6) $[\exists x \alpha(x)] := [\neg(\forall x \neg \alpha(x))]$.

(7) If $\tilde{A}, \tilde{B} \in \mathfrak{S}(X)$, then

(a) $[\tilde{A} \subseteq \tilde{B}] := [\forall x(x \in \tilde{A} \rightarrow x \in \tilde{B})] = \inf_{x \in X} \min(1, 1 - \tilde{A}(x) + \tilde{B}(x))$,

(b) $[A \equiv B] := [(\tilde{A} \subseteq \tilde{B}) \wedge (\tilde{B} \subseteq \tilde{A})]$.

Secondly, we give the following definitions which are used in the sequel.

Definition 2.1 ([2]). Let X be a universe of discourse and $\tau \in \mathfrak{S}(P(X))$ satisfy the following conditions:

(1) $\tau(X) = 1$ and $\tau(\phi) = 1$;

(2) for any A, B , $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;

(3) for any $\{A_\lambda : \lambda \in \Lambda\}$, $\tau(\bigcup_{\lambda \in \Lambda} A_\lambda) \geq \bigwedge_{\lambda \in \Lambda} \tau(A_\lambda)$.

Then τ is a fuzzifying topology and (X, τ) a fuzzifying topological space.

Definition 2.2 ([2]). Let (X, τ) be a fuzzifying topological space. Then

(1) The family of all fuzzifying closed sets is denoted by $\mathcal{F} \in \mathfrak{S}(P(X))$, and defined as follows: $A \in \mathcal{F} := X \sim A \in \tau$, where $X \sim A$ is the complement of A .

(2) The neighborhood system of $x \in X$ is denoted by $N_x \in \mathfrak{S}(P(X))$ and defined as follows: $N_x(A) = \sup_{x \in B \subseteq A} \tau(B)$.

(3) The closure $cl(A)$ of $A \subseteq X$ is defined as follows: $cl(A)(x) = 1 - N_x(X \sim A)$.

Definition 2.3 ([4]). Let (X, τ) and (Y, σ) be two fuzzifying topological spaces.

(1) A unary fuzzy predicate $C \in \mathfrak{S}(Y^X)$, called fuzzy continuity, is given as follows: $f \in C := \forall u(u \in \sigma \rightarrow f^{-1}(u) \in \tau)$. i.e., $C(f) = \inf_{u \in P(Y)} \min(1, 1 - \sigma(u) + \tau(f^{-1}(u)))$.

(2) A unary fuzzy predicate $O \in \mathfrak{S}(Y^X)$, called fuzzy openness, is given as follows: $f \in O := \forall u(u \in \tau \rightarrow f(u) \in \sigma)$. i.e., $O(f) = \inf_{u \in P(X)} \min(1, 1 - \tau(u) + \sigma(f(u)))$.

Definition 2.4 ([6]). Let (X, τ_1) and (X, τ_2) be two fuzzifying topological spaces. Then a system (X, τ_1, τ_2) consisting of a universe of discourse X with two fuzzifying topologies τ_1 and τ_2 on X is called a fuzzifying bitopological space.

Definition 2.5 ([8]). Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two fuzzifying bitopological spaces. A mapping $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be pairwise fuzzy continuous (resp. pairwise fuzzy open) if $f : (X, \tau_1) \rightarrow (Y, \sigma_1)$ and $f : (X, \tau_2) \rightarrow (Y, \sigma_2)$ are fuzzy continuous (resp. fuzzy open).

Lemma 2.1 ([9]). For any $\alpha, \beta \in [0, 1]$, $\models \alpha \rightarrow (\beta \rightarrow \alpha)$.

3. Separation Axioms in Fuzzifying Bitopological Spaces

Definition 3.1. Let (X, τ_1, τ_2) be a fuzzifying bitopological space.

(1) A set A is said to be a pairwise open if and only if $A \in \tau_1 \cap \tau_2$, i.e.,

$$\mathcal{O}^p(A) = \min(\tau_1(A), \tau_2(A)).$$

(2) A set B is said to be a pairwise closed if and only if $X \sim B \in \mathcal{O}^p$, i.e.,

$$B \in \mathcal{F}^p := X \sim B \in \mathcal{O}^p.$$

Lemma 3.1. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then

$$\mathcal{F}^p(B) = \min(\mathcal{F}_1(B), \mathcal{F}_2(B)).$$

Proof. $\mathcal{F}^p(B) = \mathcal{O}^p(X \sim B) = \min(\tau_1(X \sim B), \tau_2(X \sim B)) = \min(\mathcal{F}_1(B), \mathcal{F}_2(B))$. $\square$

Lemma 3.2. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then

$$(1) \models \mathcal{O}^p \subseteq \tau_i, i = 1, 2.$$

$$(2) \models \mathcal{F}^p \subseteq \mathcal{F}_i, i = 1, 2.$$

Proof.

$$(1) \mathcal{O}^p(A) = \min(\tau_1(A), \tau_2(A)) \leq \tau_i(A).$$

(2) It is similar to (1) above. $\square$

Remark 3.1. For simplicity we put the following notations:

$$K_{x,y}^{(i,j)} := \exists A((A \in N_x^i \wedge y \notin A) \vee (A \in N_y^j \wedge x \notin A)),$$

where N_x^i is the neighborhood system of x with respect to τ_i ,

$$H_{x,y}^{(i,j)} := \exists B \exists C (B \in N_x^i \wedge C \in N_y^j \wedge y \notin B \wedge x \notin C),$$

$$M_{x,y}^{(i,j)} := \exists B \exists C (B \in N_x^i \wedge C \in N_y^j \wedge B \cap C = \phi),$$

$$W_{A,B}^{(i,j)} := \exists U \exists V (U \in \tau_i \wedge V \in \tau_j \wedge A \subseteq V \wedge B \subseteq U \wedge U \cap V = \phi).$$

Definition 3.2. Let Ω be the class of all fuzzifying bitopological spaces. The unary fuzzy predicates $T_n^{(i,j)} \in \mathfrak{S}(\Omega)$, $n = 0, 1, 2$, $T_N^{(i,j)} \in \mathfrak{S}(\Omega)$ and $R_n^{(i,j)} \in \mathfrak{S}(\Omega)$, $n = 0, 1$ are defined as follows respectively:

$$\begin{aligned} (X, \tau_1, \tau_2) &\in T_0^{(i,j)} \\ &:= \forall x \forall y (x \in X \wedge y \in X \wedge x \neq y \rightarrow K_{x,y}^{(i,j)}), \\ (X, \tau_1, \tau_2) &\in T_1^{(i,j)} \\ &:= \forall x \forall y (x \in X \wedge y \in X \wedge x \neq y \rightarrow H_{x,y}^{(i,j)}), \\ (X, \tau_1, \tau_2) &\in T_2^{(i,j)} \\ &:= \forall x \forall y (x \in X \wedge y \in X \wedge x \neq y \rightarrow M_{x,y}^{(i,j)}), \\ (X, \tau_1, \tau_2) &\in T_N^{(i,j)} \\ &:= \forall A \forall B (A \in \mathcal{F}_i \wedge B \in \mathcal{F}_j \wedge A \cap B = \phi \rightarrow W_{A,B}^{(i,j)}), \\ (X, \tau_1, \tau_2) &\in R_0^{(i,j)} \\ &:= \forall x \forall y (x \in X \wedge y \in X \wedge x \neq y \rightarrow (K_{x,y}^{(i,j)} \rightarrow H_{x,y}^{(i,j)})), \\ (X, \tau_1, \tau_2) &\in R_1^{(i,j)} \\ &:= \forall x \forall y (x \in X \wedge y \in X \wedge x \neq y \rightarrow (K_{x,y}^{(i,j)} \rightarrow M_{x,y}^{(i,j)})). \end{aligned}$$

Remark 3.2. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Note that

$$(1) T_n^{(1,2)} = T_n^{(2,1)}, n = 0, 1, 2.$$

$$(2) T_N^{(1,2)} = T_N^{(2,1)}.$$

$$(3) R_n^{(1,2)} = R_n^{(2,1)}, n = 0, 1.$$

Definition 3.3. Let Ω be the class of all fuzzifying bitopological spaces. The unary fuzzy predicates $T_{R_i}^{(i,j)}$, $T_{R_j}^{(i,j)}$ and $T_R^{(i,j)} \in \mathfrak{S}(\Omega)$, are defined as follows:

$$(1) (X, \tau_1, \tau_2) \in T_{R_i}^{(i,j)} := \forall x \forall U (x \in X \wedge U \in \mathcal{F}_i \wedge x \notin U \rightarrow \exists A \exists B (A \in N_x^i \wedge B \in \tau_j \wedge U \subseteq B \wedge A \cap B = \phi)).$$

$$(2) (X, \tau_1, \tau_2) \in T_{R_j}^{(i,j)} := \forall x \forall U (x \in X \wedge U \in \mathcal{F}_j \wedge x \notin U \rightarrow \exists A \exists B (A \in N_x^i \wedge B \in \tau_i \wedge U \subseteq B \wedge A \cap B = \phi)).$$

$$(3) (X, \tau_1, \tau_2) \in T_R^{(i,j)} := (X, \tau_1, \tau_2) \in T_{R_i}^{(i,j)} \wedge (X, \tau_1, \tau_2) \in T_{R_j}^{(i,j)}.$$

The following example shows that generally $T_{R_i}^{(i,j)} = T_{R_j}^{(i,j)}$ need not be true.

Example 3.1. Let $X = \{a, b, c\}$ and τ_1, τ_2 be two fuzzifying topologies defined as follows:

$$\begin{aligned} \tau_1(A) &= \begin{cases} 1, & \text{if } A \in \{\phi, X, \{a, b\}\}, \\ 1/4, & \text{if } A = \{c\}, \\ 0, & \text{if } A \in \{\{a\}, \{b\}, \{a, c\}, \{b, c\}\}, \end{cases} \\ \tau_2(A) &= \begin{cases} 1, & \text{if } A \in \{\phi, X, \{a, b\}\}, \\ 1/8, & \text{if } A = \{c\}, \\ 0, & \text{if } A \in \{\{a\}, \{b\}, \{a, c\}, \{b, c\}\}. \end{cases} \end{aligned}$$

Note that

$$\begin{aligned} \mathcal{F}_1(A) &= \begin{cases} 1, & \text{if } A \in \{\phi, X, \{c\}\}, \\ 1/4, & \text{if } A = \{a, b\}, \\ 0, & \text{if } A \in \{\{b, c\}, \{a, c\}, \{b\}, \{a\}\}, \end{cases} \\ \mathcal{F}_2(A) &= \begin{cases} 1, & \text{if } A \in \{\phi, X, \{c\}\}, \\ 1/8, & \text{if } A = \{a, b\}, \\ 0, & \text{if } A \in \{\{b, c\}, \{a, c\}, \{b\}, \{a\}\}. \end{cases} \end{aligned}$$

Then we have $T_{R_1}^{(1,2)}(X, \tau_1, \tau_2) = 1/8 \neq 1/4 = T_{R_2}^{(1,2)}(X, \tau_1, \tau_2)$.

Definition 3.4. Let Ω be the class of all fuzzifying bitopological spaces. The unary fuzzy predicates $T_3^{(i,j)}$, $T_4^{(i,j)} \in \mathfrak{S}(\Omega)$ are defined as follows respectively:

$$(1) T_3^{(i,j)}(X, \tau_1, \tau_2) := T_R^{(i,j)}(X, \tau_1, \tau_2) \wedge T_1^{(i,j)}(X, \tau_1, \tau_2).$$

$$(2) T_4^{(i,j)}(X, \tau_1, \tau_2) := T_N^{(i,j)}(X, \tau_1, \tau_2) \wedge T_1^{(i,j)}(X, \tau_1, \tau_2).$$

Lemma 3.3. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then

$$(1) \models M_{x,y}^{(i,j)} \rightarrow H_{x,y}^{(i,j)}.$$

$$(2) \models H_{x,y}^{(i,j)} \rightarrow K_{x,y}^{(i,j)}.$$

$$(3) \models M_{x,y}^{(i,j)} \rightarrow K_{x,y}^{(i,j)}.$$

Proof. The proof is obvious. $\square$

Theorem 3.1. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then

- (1) $\models (X, \tau_1, \tau_2) \in T_1^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in T_0^{(i,j)}.$
- (2) $\models (X, \tau_1, \tau_2) \in T_2^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in T_1^{(i,j)}.$
- (3) $\models (X, \tau_1, \tau_2) \in T_2^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in T_0^{(i,j)}.$

Proof. From Lemma 3.3, it is clear. $\square$

Theorem 3.2. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then

- (1) $\models (X, \tau_1, \tau_2) \in R_1^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in R_0^{(i,j)}.$
- (2) $\models (X, \tau_1, \tau_2) \in T_1^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in R_0^{(i,j)}.$
- (3) $\models (X, \tau_1, \tau_2) \in T_2^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in R_0^{(i,j)}.$
- (4) $\models (X, \tau_1, \tau_2) \in T_2^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in R_1^{(i,j)}.$

Proof. From Lemma 2.1, Lemma 3.3 and Theorem 3.1, the proof becomes obvious. $\square$

Theorem 3.3. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then

$$\models (X, \tau_1, \tau_2) \in T_0^{(i,j)} \leftrightarrow \forall x \forall y (x \in X \wedge y \in X \wedge x \neq y \rightarrow x \notin cl_i(\{y\}) \vee y \notin cl_j(\{x\})).$$

Proof. The proof is obvious. $\square$

Theorem 3.4. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then

- (1) $\models (X, \tau_1, \tau_2) \in T_1^{(i,j)} \rightarrow \forall x (\{x\} \in \mathcal{F}_i).$
- (2) $\models (X, \tau_1, \tau_2) \in T_1^{(i,j)} \rightarrow \forall x (\{x\} \in \mathcal{F}_j).$

Proof.

$$\begin{aligned} (1) \quad & T_1^{(i,j)}(X, \tau_1, \tau_2) \\ &= \inf_{x_1 \neq x_2} \min(\sup_{x_2 \notin A} N_{x_1}^i(A), \sup_{x_1 \notin B} N_{x_2}^j(B)) \\ &= \inf_{x_1 \neq x_2} \min(N_{x_1}^i(X \sim \{x_2\}), N_{x_2}^j(X \sim \{x_1\})) \\ &\leq \inf_{x_1 \neq x_2} N_{x_1}^i(X \sim \{x_2\}) \\ &= \inf_{x_2 \in X} \inf_{x_1 \in X \sim \{x_2\}} N_{x_1}^i(X \sim \{x_2\}) \\ &= \inf_{x_2 \in X} \tau_i(X \sim \{x_2\}) \\ &= \inf_{x \in X} \tau_i(X \sim \{x\}) = \inf_{x \in X} \mathcal{F}_i(\{x\}). \end{aligned}$$

(2) It is similar to (1) above. $\square$

The following examples show that generally the reverse of Theorem 3.4 need not be true.

Example 3.2. Let $X = \{a, b\}$ and τ_1, τ_2 be two fuzzifying topologies defined as follows:

$$\tau_1(A) = \begin{cases} 1, & \text{if } A \in \{\phi, X\}, \\ 1/5, & \text{if } A = \{a\}, \\ 1/2, & \text{if } A = \{b\}, \end{cases}$$

$$\tau_2(A) = \begin{cases} 1, & \text{if } A \in \{\phi, X\}, \\ 1/4, & \text{if } A = \{a\}, \\ 1/8, & \text{if } A = \{b\}. \end{cases}$$

Note that

$$\mathcal{F}_1(A) = \begin{cases} 1, & \text{if } A \in \{\phi, X\}, \\ 1/2, & \text{if } A = \{a\}, \\ 1/5, & \text{if } A = \{b\}, \end{cases}$$

$$\mathcal{F}_2(A) = \begin{cases} 1, & \text{if } A \in \{\phi, X\}, \\ 1/8, & \text{if } A = \{a\}, \\ 1/4, & \text{if } A = \{b\}. \end{cases}$$

Note that $[\forall x (\{x\} \in \mathcal{F}_1)] = 1/5 \not\leq 1/8 = [(X, \tau_1, \tau_2) \in T_1^{(i,j)}].$

Example 3.3. Let $X = \{a, b\}$ and τ_1, τ_2 be two fuzzifying topologies defined as follows:

$$\tau_1(A) = \begin{cases} 1, & \text{if } A \in \{\phi, X\}, \\ 1/6, & \text{if } A = \{a\}, \\ 1/3, & \text{if } A = \{b\}, \end{cases}$$

$$\tau_2(A) = \begin{cases} 1, & \text{if } A \in \{\phi, X\}, \\ 1/4, & \text{if } A = \{a\}, \\ 1/2, & \text{if } A = \{b\}. \end{cases}$$

Note that

$$\mathcal{F}_1(A) = \begin{cases} 1, & \text{if } A \in \{\phi, X\}, \\ 1/3, & \text{if } A = \{a\}, \\ 1/6, & \text{if } A = \{b\}, \end{cases}$$

$$\mathcal{F}_2(A) = \begin{cases} 1, & \text{if } A \in \{\phi, X\}, \\ 1/2, & \text{if } A = \{a\}, \\ 1/4, & \text{if } A = \{b\}. \end{cases}$$

Note that $[\forall x (\{x\} \in \mathcal{F}_2)] = 1/4 \not\leq 1/6 = [(X, \tau_1, \tau_2) \in T_1^{(i,j)}].$

Theorem 3.5. Let (X, τ_1, τ_2) be a fuzzifying bitopological

space. Then

$$\models (X, \tau_1, \tau_2) \in T_1^{(i,j)} \leftrightarrow \forall x(\{x\} \in \mathcal{F}^p).$$

Proof. For any $x_1, x_2 \in X$ with $x_1 \neq x_2$.

$$\begin{aligned} [\forall x(\{x\} \in \mathcal{F}^p)] &= \inf_{x \in X} [\{x\} \in \mathcal{F}^p] \\ &= \inf_{x \in X} \min([\{x\} \in \mathcal{F}_i], [\{x\} \in \mathcal{F}_j]) \\ &\leq \inf_{x \in X} [\{x\} \in \mathcal{F}_i] = \inf_{x \in X} \tau_i(X \sim \{x\}) \\ &= \inf_{x \in X} \inf_{y \in X \sim \{x\}} N_y^i(X \sim \{x\}) \\ &\leq \inf_{y \in X \sim \{x_2\}} N_y^i(X \sim \{x_2\}) \\ &\leq N_{x_1}^i(X \sim \{x_2\}) = \sup_{x_2 \notin A} N_{x_1}^i(A). \end{aligned}$$

By the same way, we have

$$[\forall x(\{x\} \in \mathcal{F}^p)] \leq \sup_{x_1 \notin B} N_{x_2}^j(B).$$

So

$$\begin{aligned} [\forall x(\{x\} \in \mathcal{F}^p)] &\leq \inf_{x_1 \neq x_2} \min(\sup_{x_2 \notin A} N_{x_1}^i(A), \sup_{x_1 \notin B} N_{x_2}^j(B)) \\ &= T_1^{(i,j)}(X, \tau_1, \tau_2). \end{aligned}$$

On the other hand, from Theorem 3.4 we have

$$\begin{aligned} T_1^{(i,j)}(X, \tau_1, \tau_2) &\leq \inf_{x \in X} \min(\mathcal{F}_i(\{x\}), \mathcal{F}_j(\{x\})) \\ &= [\forall x(\{x\} \in \mathcal{F}^p)]. \end{aligned}$$

Therefore, $T_1^{(i,j)}(X, \tau_1, \tau_2) = [\forall x(\{x\} \in \mathcal{F}^p)]$. $\square$

Definition 3.5. Let (X, τ_1, τ_2) be a fuzzifying bitopological space, we define

- (1) ${}^1T_{R_i}^{(i,j)}(X, \tau_1, \tau_2) := \forall x \forall U (x \in X \wedge U \in \mathcal{F}_i \wedge x \notin U \rightarrow \exists A (A \in N_x^i \wedge cl_j(A) \cap U = \phi))$.
- (2) ${}^1T_{R_j}^{(i,j)}(X, \tau_1, \tau_2) := \forall x \forall U (x \in X \wedge U \in \mathcal{F}_j \wedge x \notin U \rightarrow \exists A (A \in N_x^j \wedge cl_i(A) \cap U = \phi))$.
- (3) ${}^1T_R^{(i,j)} := {}^1T_{R_i}^{(i,j)} \wedge {}^1T_{R_j}^{(i,j)}$.
- (4) ${}^2T_{R_i}^{(i,j)}(X, \tau_1, \tau_2) := \forall x \forall V (x \in X \wedge V \in \tau_i \wedge x \in V \rightarrow \exists B (B \in N_x^i \wedge cl_j(B) \subseteq V))$.
- (5) ${}^2T_{R_j}^{(i,j)}(X, \tau_1, \tau_2) := \forall x \forall V (x \in X \wedge V \in \tau_j \wedge x \in V \rightarrow \exists B (B \in N_x^j \wedge cl_i(B) \subseteq V))$.
- (6) ${}^2T_R^{(i,j)} := {}^2T_{R_i}^{(i,j)} \wedge {}^2T_{R_j}^{(i,j)}$.

Theorem 3.6. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then

- (1) $\models T_{R_i}^{(i,j)}(X, \tau_1, \tau_2) \leftrightarrow {}^nT_{R_i}^{(i,j)}(X, \tau_1, \tau_2), n = 1, 2$.
- (2) $\models T_{R_j}^{(i,j)}(X, \tau_1, \tau_2) \leftrightarrow {}^nT_{R_j}^{(i,j)}(X, \tau_1, \tau_2), n = 1, 2$.
- (3) $\models T_R^{(i,j)}(X, \tau_1, \tau_2) \leftrightarrow {}^nT_R^{(i,j)}(X, \tau_1, \tau_2), n = 1, 2$.

Proof. It is similar to proof of Theorem 2.6 in [5]. $\square$

Theorem 3.7. Let (X, τ_1, τ_2) be a fuzzifying bitopological space, and let

- (1) ${}^1T_N^{(i,j)}(X, \tau_1, \tau_2) := \forall A \forall B (A \in \tau_i \wedge B \in \mathcal{F}_j \wedge B \subseteq A \rightarrow \exists U \exists V (U \in \tau_i \wedge V \in \mathcal{F}_j \wedge B \subseteq U \subseteq V \subseteq A))$.
- (2) ${}^2T_N^{(i,j)}(X, \tau_1, \tau_2) := \forall A \forall B (A \in \mathcal{F}_i \wedge B \in \mathcal{F}_j \wedge A \cap B = \phi \rightarrow \exists U (U \in \tau_i \wedge A \subseteq U \wedge cl_j(U) \cap B = \phi))$.
- (3) ${}^3T_N^{(i,j)}(X, \tau_1, \tau_2) := \forall A \forall B (A \in \mathcal{F}_i \wedge B \in \tau_j \wedge A \subseteq B \rightarrow \exists U (U \in \tau_i \wedge A \subseteq U \wedge cl_j(U) \subseteq B))$.

Then $\models T_N^{(i,j)}(X, \tau_1, \tau_2) \leftrightarrow {}^nT_N^{(i,j)}(X, \tau_1, \tau_2), n = 1, 2, 3$.

Proof.

$$\begin{aligned} (1) T_N^{(i,j)}(X, \tau_1, \tau_2) &= \inf_{A \cap B = \phi} \min(1, 1 - \min(\mathcal{F}_i(A), \mathcal{F}_j(B))) \\ &\quad + \sup_{U \cap V = \phi, A \subseteq V, B \subseteq U} \min(\tau_i(U), \tau_j(V)) \\ &= \inf_{X \sim A \cap B = \phi} \min(1, 1 - \min(\mathcal{F}_i(X \sim A), \mathcal{F}_j(B))) \\ &\quad + \sup_{U \cap X \sim V = \phi, X \sim A \subseteq X \sim V, B \subseteq U} \min(\tau_i(U), \tau_j(X \sim V)) \\ &= \inf_{B \subseteq A} \min(1, 1 - \min(\tau_i(A), \mathcal{F}_j(B))) \\ &\quad + \sup_{U \subseteq V, V \subseteq A, B \subseteq U} \min(\tau_i(U), \mathcal{F}_j(V)) \\ &= \inf_{B \subseteq A} \min(1, 1 - \min(\tau_i(A), \mathcal{F}_j(B))) \\ &\quad + \sup_{B \subseteq U \subseteq V \subseteq A} \min(\tau_i(U), \mathcal{F}_j(V)) = {}^1T_N^{(i,j)}(X, \tau_1, \tau_2). \end{aligned}$$

(2) and (3) are similar to that of Theorem 3.6. $\square$

Now we define a new weaker form of pairwise normality in the fuzzifying bitopological spaces.

Definition 3.6. Let Ω be the class of all fuzzifying bitopological spaces. The unary fuzzy predicates ${}^wT_N^{(i,j)} \in \mathfrak{S}(\Omega)$ defined as follows:

$$\begin{aligned} (X, \tau_1, \tau_2) \in {}^wT_N^{(i,j)} \\ := \forall A \forall B (A \in \mathcal{F}^p \wedge B \in \mathcal{F}^p \wedge A \cap B = \phi \rightarrow W_{A,B}^{(i,j)}). \end{aligned}$$

Theorem 3.8. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then

$$\models (X, \tau_1, \tau_2) \in T_N^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in {}^w T_N^{(i,j)}.$$

Proof. It is obtained from part (2) of Lemma 3.2. $\square$

The following example shows that generally the reverse of Theorem 3.8 need not be true.

Example 3.4. Let $X = \{a, b, c, d\}$, τ_1, τ_2 be two fuzzifying topologies defined as follows:

$$\tau_1(A) = \begin{cases} 1, & \text{if } A \in \{\phi, X, \{a, b, d\}\}, \\ 1/2, & \text{if } A = \{c\}, \\ 0, & \text{if } o.w., \end{cases}$$

$$\tau_2(A) = \begin{cases} 1, & \text{if } A \in \{\phi, X, \{b, c, d\}\}, \\ 1/4, & \text{if } A = \{a\}, \\ 0, & \text{if } o.w.. \end{cases}$$

Note that

$$\mathcal{O}^p(A) = \begin{cases} 1, & \text{if } A \in \{\phi, X\}, \\ 0, & \text{if } o.w., \end{cases}$$

$$\mathcal{F}^p(A) = \begin{cases} 1, & \text{if } A \in \{\phi, X\}, \\ 0, & \text{if } o.w., \end{cases}$$

and

$$\mathcal{F}_1(A) = \begin{cases} 1, & \text{if } A \in \{\phi, X, \{c\}\}, \\ 1/2, & \text{if } A = \{a, b, d\}, \\ 0, & \text{if } o.w., \end{cases}$$

$$\mathcal{F}_2(A) = \begin{cases} 1, & \text{if } A \in \{\phi, X, \{a\}\}, \\ 1/4, & \text{if } A = \{b, c, d\}, \\ 0, & \text{if } o.w.. \end{cases}$$

Then we have $[(X, \tau_1, \tau_2) \in {}^w T_N^{(1,2)}] = 1 \not\leq 0 = [(X, \tau_1, \tau_2) \in T_N^{(1,2)}]$.

4. Relation among Separation Axioms in Fuzzifying Bitopological Spaces

Theorem 4.1. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then

$$(1) \models T_{R_i}^{(i,j)}(X, \tau_1, \tau_2) \dot{\wedge} T_1^{(i,j)}(X, \tau_1, \tau_2) \rightarrow T_2^{(i,j)}(X, \tau_1, \tau_2).$$

$$(2) \models T_{R_j}^{(i,j)}(X, \tau_1, \tau_2) \dot{\wedge} T_1^{(i,j)}(X, \tau_1, \tau_2) \rightarrow T_2^{(i,j)}(X, \tau_1, \tau_2).$$

Proof. From Theorem 3.5, the proof becomes obvious. $\square$

Theorem 4.2. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then

$$(1) \models T_4^{(i,j)}(X, \tau_1, \tau_2) \rightarrow T_{R_i}^{(i,j)}(X, \tau_1, \tau_2).$$

$$(2) \models T_4^{(i,j)}(X, \tau_1, \tau_2) \rightarrow T_{R_j}^{(i,j)}(X, \tau_1, \tau_2).$$

Proof.

(1) Since $T_4^{(i,j)}(X, \tau_1, \tau_2) = \max(0, T_N^{(i,j)}(X, \tau_1, \tau_2) + T_1^{(i,j)}(X, \tau_1, \tau_2) - 1)$, then we prove that

$$T_{R_i}^{(i,j)}(X, \tau_1, \tau_2) \geq T_N^{(i,j)}(X, \tau_1, \tau_2) + T_1^{(i,j)}(X, \tau_1, \tau_2) - 1.$$

In fact,

$$\begin{aligned} & T_N^{(i,j)}(X, \tau_1, \tau_2) + T_1^{(i,j)}(X, \tau_1, \tau_2) \\ &= \inf_{U \cap V = \phi} \min \left(1, 1 - \min(\tau_i(X \sim U), \tau_j(X \sim V)) \right) \\ & \quad + \sup_{A \cap B = \phi, U \subseteq B, V \subseteq A} \min(\tau_i(A), \tau_j(B)) \\ & \quad + \inf_{z \in X} \min(\tau_i(X \sim \{z\}), \tau_j(X \sim \{z\})) \\ &\leq \inf_{x \notin U} \min \left(1, 1 - \min(\tau_i(X \sim U), \tau_j(X \sim \{x\})) \right) \\ & \quad + \sup_{A \cap B = \phi, U \subseteq B} \min(N_x^i(A), \tau_j(B)) \\ & \quad + \inf_{z \in X} \tau_j(X \sim \{z\}) \\ &= \inf_{x \notin U} \min \left(1, \max \left(1 - \tau_i(X \sim U) \right. \right. \\ & \quad + \sup_{A \cap B = \phi, U \subseteq B} \min(N_x^i(A), \tau_j(B)), 1 - \tau_j(X \sim \{x\}) \\ & \quad + \sup_{A \cap B = \phi, U \subseteq B} \min(N_x^i(A), \tau_j(B)) \left. \right) \\ & \quad + \inf_{z \in X} \tau_j(X \sim \{z\}) \\ &= \inf_{x \notin U} \max \left(\min \left(1, 1 - \tau_i(X \sim U) \right. \right. \\ & \quad + \sup_{A \cap B = \phi, U \subseteq B} \min(N_x^i(A), \tau_j(B)) \left. \right), \\ & \quad \min \left(1, 1 - \tau_j(X \sim \{x\}) \right. \\ & \quad + \sup_{A \cap B = \phi, U \subseteq B} \min(N_x^i(A), \tau_j(B)) \left. \right) \\ & \quad + \inf_{z \in X} \tau_j(X \sim \{z\}) \left. \right) \\ &\leq \inf_{x \notin U} \max \left(\min \left(1, 1 - \tau_i(X \sim U) \right. \right. \\ & \quad + \sup_{A \cap B = \phi, U \subseteq B} \min(N_x^i(A), \tau_j(B)) \left. \right) + \tau_j(X \sim \{x\}), \\ & \quad \min \left(1, 1 - \tau_j(X \sim \{x\}) \right. \end{aligned}$$

$$\begin{aligned}
 & + \sup_{A \cap B = \phi, U \subseteq B} \min(N_x^i(A), \tau_j(B)) + \tau_j(X \sim \{x\}) \\
 & \leq \inf_{x \notin U} \max \left(\min \left(1, 1 - \tau_i(X \sim U) \right. \right. \\
 & \quad \left. \left. + \sup_{A \cap B = \phi, U \subseteq B} \min(N_x^i(A), \tau_j(B)) \right) + \tau_j(X \sim \{x\}), \right. \\
 & \quad \left. 1 + \sup_{A \cap B = \phi, U \subseteq B} \min(N_x^i(A), \tau_j(B)) \right) \\
 & \leq \inf_{x \notin U} \min \left(1, 1 - \tau_i(X \sim U) \right. \\
 & \quad \left. + \sup_{A \cap B = \phi, U \subseteq B} \min(N_x^i(A), \tau_j(B)) \right) + 1 \\
 & = T_{R_i}^{(i,j)}(X, \tau_1, \tau_2) + 1.
 \end{aligned}$$

(2) It is similar to (1) above. $\square$

From Theorem 4.1 and Theorem 4.2, we have the following result:

Corollary 4.1. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then

$$\begin{aligned}
 (1) & \models T_3^{(i,j)}(X, \tau_1, \tau_2) \rightarrow T_2^{(i,j)}(X, \tau_1, \tau_2). \\
 (2) & \models T_4^{(i,j)}(X, \tau_1, \tau_2) \rightarrow T_R^{(i,j)}(X, \tau_1, \tau_2).
 \end{aligned}$$

The following example shows that generally $T_4^{(i,j)}(X, \tau_1, \tau_2) \rightarrow T_3^{(i,j)}(X, \tau_1, \tau_2)$ need not be true.

Example 4.1. For $X = \{a, b\}$, let τ_1 and τ_1 be two fuzzifying topologies, which are defined on X in Example 3.2, then we have $T_4^{(i,j)}(X, \tau_1, \tau_2) = 1/8 \not\leq 0 = T_3^{(i,j)}(X, \tau_1, \tau_2)$.

Theorem 4.3. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then

$$\begin{aligned}
 (1) & \models (X, \tau_1, \tau_2) \in T_1^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in R_0^{(i,j)} \wedge \\
 & (X, \tau_1, \tau_2) \in T_0^{(i,j)}. \\
 (2) & \text{ If } T_0^{(i,j)}(X, \tau_1, \tau_2) = 1, \text{ then} \\
 & \models (X, \tau_1, \tau_2) \in T_1^{(i,j)} \leftrightarrow (X, \tau_1, \tau_2) \in R_0^{(i,j)} \\
 & \wedge (X, \tau_1, \tau_2) \in T_0^{(i,j)}.
 \end{aligned}$$

Proof.

(1) It is obtained from part (1) of the Theorem 3.1 and part (2) Theorem 3.2.

(2) Since $T_0^{(i,j)}(X, \tau_1, \tau_2) = 1$, then for every $x, y \in X$ such that $x \neq y$, we have $[K_{x,y}^{(i,j)}] = 1$. So

$$\begin{aligned}
 & R_0^{(i,j)}(X, \tau_1, \tau_2) \wedge T_0^{(i,j)}(X, \tau_1, \tau_2) \\
 & = R_0^{(i,j)}(X, \tau_1, \tau_2)
 \end{aligned}$$

$$\begin{aligned}
 & = \inf_{x \neq y} \min(1, 1 - [K_{x,y}^{(i,j)}] + [H_{x,y}^{(i,j)}]) \\
 & = \inf_{x \neq y} [H_{x,y}^{(i,j)}] = T_1^{(i,j)}(X, \tau_1, \tau_2). \quad \square
 \end{aligned}$$

Theorem 4.4. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then

$$(1) \models (X, \tau_1, \tau_2) \in T_2^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in R_1^{(i,j)} \wedge (X, \tau_1, \tau_2) \in T_0^{(i,j)}.$$

(2) If $T_0^{(i,j)}(X, \tau_1, \tau_2) = 1$, then

$$\begin{aligned}
 & \models (X, \tau_1, \tau_2) \in T_2^{(i,j)} \leftrightarrow (X, \tau_1, \tau_2) \in R_1^{(i,j)} \\
 & \wedge (X, \tau_1, \tau_2) \in T_0^{(i,j)}.
 \end{aligned}$$

Proof.

(1) It is obtained from part (3) of the Theorem 3.1 and part (4) of the Theorem 3.2.

(2) It is similar to proof of part (2) of the Theorem 4.3. $\square$

Remark 4.1. In the crisp setting, i.e, if the underlying fuzzifying bitopology is the ordinary bitopology one can have that

$$\begin{aligned}
 (1) & \models (X, \tau_1, \tau_2) \in T_1^{(i,j)} \leftrightarrow (X, \tau_1, \tau_2) \in R_0^{(i,j)} \\
 & \wedge (X, \tau_1, \tau_2) \in T_0^{(i,j)}. \\
 (2) & \models (X, \tau_1, \tau_2) \in T_2^{(i,j)} \leftrightarrow (X, \tau_1, \tau_2) \in R_1^{(i,j)} \\
 & \wedge (X, \tau_1, \tau_2) \in T_0^{(i,j)}.
 \end{aligned}$$

Generally these statements may not be true in fuzzifying bitopology as illustrated by the following example.

Example 4.2. Let $X = \{a, b\}$ and τ_1, τ_2 be two fuzzifying topologies defined as follows:

$$\begin{aligned}
 \tau_1(A) & = \begin{cases} 1, & \text{if } A \in \{\phi, X\}, \\ 1/4, & \text{if } A = \{a\}, \\ 1/5, & \text{if } A = \{b\}, \end{cases} \\
 \tau_2(A) & = \begin{cases} 1, & \text{if } A \in \{\phi, X\}, \\ 1/3, & \text{if } A = \{a\}, \\ 1/2, & \text{if } A = \{b\}. \end{cases}
 \end{aligned}$$

Note that $T_0^{(i,j)}(X, \tau_1, \tau_2) = 1/3$, $T_1^{(i,j)}(X, \tau_1, \tau_2) = 1/5 = T_2^{(i,j)}(X, \tau_1, \tau_2)$ and $R_0^{(i,j)}(X, \tau_1, \tau_2) = 3/4 = R_1^{(i,j)}(X, \tau_1, \tau_2)$. Hence,

$$\begin{aligned}
 & R_0^{(i,j)}(X, \tau_1, \tau_2) \wedge T_0^{(i,j)}(X, \tau_1, \tau_2) = 1/3 \neq 1/5 \\
 & = T_1^{(i,j)}(X, \tau_1, \tau_2), \\
 & R_1^{(i,j)}(X, \tau_1, \tau_2) \wedge T_0^{(i,j)}(X, \tau_1, \tau_2) = 1/3 \neq 1/5
 \end{aligned}$$

$$= T_2^{(i,j)}(X, \tau_1, \tau_2).$$

Theorem 4.5. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then

- (1) $(X, \tau_1, \tau_2) \in R_0^{(i,j)} \wedge (X, \tau_1, \tau_2) \in T_0^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in T_1^{(i,j)}.$
- (2) If $T_0^{(i,j)}(X, \tau_1, \tau_2) = 1$, then
 $\models (X, \tau_1, \tau_2) \in R_0^{(i,j)} \wedge (X, \tau_1, \tau_2) \in T_0^{(i,j)} \leftrightarrow (X, \tau_1, \tau_2) \in T_1^{(i,j)}.$

Proof.

- (1) It is clear.
- (2) It is similar to the proof of part (2) of the Theorem 4.3. $\square$

Theorem 4.6. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then

- (1) $\models (X, \tau_1, \tau_2) \in R_1^{(i,j)} \wedge (X, \tau_1, \tau_2) \in T_0^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in T_2^{(i,j)}.$
- (2) If $T_0^{(i,j)}(X, \tau_1, \tau_2) = 1$, then
 $\models (X, \tau_1, \tau_2) \in R_1^{(i,j)} \wedge (X, \tau_1, \tau_2) \in T_0^{(i,j)}$
 $\leftrightarrow (X, \tau_1, \tau_2) \in T_2^{(i,j)}.$

Proof.

- (1) It is clear.
- (2) It is similar to the proof of part (2) of the Theorem 4.3. $\square$

Theorem 4.7. Let (X, τ_1, τ_2) be a fuzzifying bitopological space. Then

- (1) $\models (X, \tau_1, \tau_2) \in T_0^{(i,j)} \rightarrow ((X, \tau_1, \tau_2) \in R_0^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in T_1^{(i,j)}).$
- (2) $\models (X, \tau_1, \tau_2) \in R_0^{(i,j)} \rightarrow ((X, \tau_1, \tau_2) \in T_0^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in T_1^{(i,j)}).$
- (3) $\models (X, \tau_1, \tau_2) \in T_0^{(i,j)} \rightarrow ((X, \tau_1, \tau_2) \in R_1^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in T_2^{(i,j)}).$
- (4) $\models (X, \tau_1, \tau_2) \in R_1^{(i,j)} \rightarrow ((X, \tau_1, \tau_2) \in T_0^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in T_2^{(i,j)}).$

Proof. From part (1) of the Theorem 3.1, part (2) of the Theorem 3.2 and part (3) of the Theorem 4.3, the proof becomes obvious. $\square$

5. Separation Axioms and Continuity in Fuzzifying Bitopological Spaces

Theorem 5.1. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two fuzzifying bitopological spaces. If a mapping $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is injective and pairwise fuzzy open with degree one, then

- (1) $\models (X, \tau_1, \tau_2) \in T_0^{(i,j)} \rightarrow (Y, \sigma_1, \sigma_2) \in T_0^{(i,j)};$
- (2) $\models (X, \tau_1, \tau_2) \in T_1^{(i,j)} \rightarrow (Y, \sigma_1, \sigma_2) \in T_1^{(i,j)};$
- (3) $\models (X, \tau_1, \tau_2) \in T_2^{(i,j)} \rightarrow (Y, \sigma_1, \sigma_2) \in T_2^{(i,j)}.$

Proof.

(1) From part (1) of the Theorem 3.2 in [8], $[f \in O_1] = 1$ and $[f \in O_2] = 1$, we have for every $v \in P(Y)$ and $x \in X$, $N_x^t(f^{-1}(V)) \leq N_{f(x)}^t(V)$, $t = 1, 2$. Therefore,

$$\begin{aligned} T_0^{(i,j)}(X, \tau_1, \tau_2) &= \inf_{x \neq y} \max(\sup_{y \notin A} N_x^i(A), \sup_{x \notin A} N_y^j(A)) \\ &= \inf_{x \neq y} \max(\sup_{y \notin A} N_x^i(f^{-1}f(A)), \sup_{x \notin A} N_y^j(f^{-1}f(A))) \\ &\leq \inf_{x \neq y} \max(\sup_{y \notin A} N_{f(x)}^i(f(A)), \sup_{x \notin A} N_{f(y)}^j(f(A))) \\ &= \inf_{f(x) \neq f(y)} \max(\sup_{f(y) \notin f(A)} N_{f(x)}^i(f(A)), \sup_{f(x) \notin f(A)} N_{f(y)}^j(f(A))) \\ &= \inf_{z \neq w} \max(\sup_{w \notin H} N_z^i(H), \sup_{z \notin H} N_w^j(H)) \\ &= T_0^{(i,j)}(Y, \sigma_1, \sigma_2). \end{aligned}$$

The proof of (2) and (3) is similar to (1) above. $\square$

Theorem 5.2. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two fuzzifying bitopological spaces. If a mapping $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is injective and pairwise fuzzy continuous with degree one, then

- (1) $\models (Y, \sigma_1, \sigma_2) \in T_0^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in T_0^{(i,j)};$
- (2) $\models (Y, \sigma_1, \sigma_2) \in T_1^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in T_1^{(i,j)};$
- (3) $\models (Y, \sigma_1, \sigma_2) \in T_2^{(i,j)} \rightarrow (X, \tau_1, \tau_2) \in T_2^{(i,j)}.$

Proof. From part (3) of the Theorem 2.1 in [4] the proof becomes obvious. $\square$

Theorem 5.3. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two fuzzifying bitopological spaces. If a mapping $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is bijective, pairwise fuzzy open and pairwise fuzzy

continuous with degree one, then

$$\begin{aligned} (1) & \models (X, \tau_1, \tau_2) \in T_N^{(i,j)} \leftrightarrow (Y, \sigma_1, \sigma_2) \in T_N^{(i,j)}; \\ (2) & \models (X, \tau_1, \tau_2) \in T_R^{(i,j)} \leftrightarrow (Y, \sigma_1, \sigma_2) \in T_R^{(i,j)}. \end{aligned}$$

Proof.

(1) (a) Since f is injective, pairwise open and pairwise continuous with degree one, we have

$$\begin{aligned} [T_N^{(i,j)}(X, \tau_1, \tau_2)] &= \inf_{A \cap B = \phi} \min(1, 1 - \min(\mathcal{F}_i(A), \mathcal{F}_j(B))) \\ &+ \sup_{U \cap V = \phi, A \subseteq V, B \subseteq U} \min(\tau_i(U), \tau_j(V)) \\ &= \inf_{A \cap B = \phi} \min(1, 1 - \min(\tau_i(X \sim A), \tau_j(X \sim B))) \\ &+ \sup_{U \cap V = \phi, A \subseteq V, B \subseteq U} \min(\tau_i(U), \tau_j(V)) \\ &\geq \inf_{A \cap B = \phi} \min(1, 1 - \min(\sigma_i(f(X \sim A)), \sigma_j(f(X \sim B)))) \\ &+ \sup_{U \cap V = \phi, A \subseteq V, B \subseteq U} \min(\tau_i(f^{-1}f(U)), \tau_j(f^{-1}f(V))) \\ &\geq \inf_{A \cap B = \phi} \min(1, 1 - \min(\sigma_i(Y \sim f(A)), \sigma_j(Y \sim f(B)))) \\ &+ \sup_{U \cap V = \phi, A \subseteq V, B \subseteq U} \min(\sigma_i(f(U)), \sigma_j(f(V))) \\ &= \inf_{H \cap G = \phi} \min(1, 1 - \min(\sigma_i(Y \sim H), \sigma_j(Y \sim G))) \\ &+ \sup_{M \cap N = \phi, H \subseteq N, G \subseteq M} \min(\sigma_i(M), \sigma_j(N)) \\ &= [T_N^{(i,j)}(Y, \sigma_1, \sigma_2)]. \end{aligned}$$

(b) Since f is surjective, pairwise open and pairwise continuous with degree one, we have

$$\begin{aligned} [T_N^{(i,j)}(Y, \sigma_1, \sigma_2)] &= \inf_{H \cap G = \phi} \min(1, 1 - \min(\sigma_i(Y \sim H), \sigma_j(Y \sim G))) \\ &+ \sup_{M \cap N = \phi, H \subseteq N, G \subseteq M} \min(\sigma_i(M), \sigma_j(N)) \\ &\geq \inf_{H \cap G = \phi} \min(1, 1 - \min(\tau_i(f^{-1}(Y \sim H)), \\ &\tau_j(f^{-1}(Y \sim G))) \\ &+ \sup_{M \cap N = \phi, H \subseteq N, G \subseteq M} \min(\sigma_i(f f^{-1}(M)), \\ &\sigma_j(f f^{-1}(N))) \\ &\geq \inf_{H \cap G = \phi} \min(1, 1 - \min(\tau_i(X \sim f^{-1}(H)), \\ &\tau_j(Y \sim f^{-1}(G))) \\ &+ \sup_{M \cap N = \phi, H \subseteq N, G \subseteq M} \min(\tau_i(f^{-1}(M)), \tau_j(f^{-1}(N))) \end{aligned}$$

$$\begin{aligned} &= \inf_{A \cap B = \phi} \min(1, 1 - \min(\tau_i(X \sim A), \tau_j(X \sim B))) \\ &+ \sup_{U \cap V = \phi, A \subseteq V, B \subseteq U} \min(\tau_i(U), \tau_j(V)) \\ &= [T_N^{(i,j)}(X, \tau_1, \tau_2)]. \end{aligned}$$

From (a) and (b), we have $[T_N^{(i,j)}(X, \tau_1, \tau_2)] = [T_N^{(i,j)}(Y, \sigma_1, \sigma_2)]$.

(2) It is similar to (1) above. $\square$

6. Conclusions

In the present paper we used Łukasiewicz fuzzy logic to extend the notions of separation axioms from the framework of fuzzifying topological spaces into the framework of fuzzifying bitopological spaces and study some relations between them. Also we investigate the image of these kinds of fuzzifying bitopological spaces under some types of fuzzy mappings.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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