Notes on Independence of Attributes in Soft Contexts

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Abstract

For the purpose of studying the formal concepts and the reduction in a formal context, we have combined the formal contexts with the soft sets to form soft contexts and proposed the soft concept in a soft context. As a series of studies, it is necessary to investigate specific properties of attributes. For this purpose, we introduce and study the notion of independent and dependent attributes in a given soft context. In particular, we will study the following: (1) Every dependent attribute is generated by some independent attributes; (2) The set of all soft concepts can be completely constructed by independent attributes.

Keywords: Formal context, Formal concept, Concept lattice, Soft set, Soft context, Soft concept, Independent attributes

1. Introduction

For the purpose of the study of hierarchical structures based on a binary relation between objects and attributes, Wille introduced FCA (formal concept analysis) [1] in 1982 and investigated the notions of context, formal concept, and concept lattice. A formal context, a type of information system, is represented in a tabular form of an object-attribute value relationship [2–5]. A formal concept is presented in pairs consisting of objects and attributes. The order relationship between two formal concepts is well defined, and it is well known that through this order relationship the collection of all formal concepts is a complete lattice. It is simply called the concept lattice [5]. Formal concept analysis has been widely applied to many information systems research fields, and many studies are actively conducted to apply problems in real-world situations. [5–11].

The concept of soft sets was introduced by Molodtsov in 1999 [12] for the purpose of dealing with complex problems and uncertainties: Let U be an initial universe set (simply, universe set) and A be a collection of characteristics or properties of objects in U . A pair (F, A) is called a *soft set* over U if F is a mapping of A into the set of all subsets of the set U . For the soft set theory, Ali et al. [13] proposed new operations that complements the concepts defined by Maji in [14].

In [15], we constructed the soft context combining the notions of formal contexts and soft sets as set-valued mappings. Additionally, we introduced and investigated the notions of soft concepts and soft concepts lattice that are closely related to formal concepts and concept lattices in FCA.

As a series of studies in [15], we would like to investigate the specific properties of attributes. In this paper, we introduce and study the notions of independent and dependent attributes in a

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given soft context. In particular, we will show the following two facts: 1) Every dependent attribute is generated by some independent attributes: 2) The set of all soft concepts can be completely constructed by independent attributes.

2. Preliminaries

We recall some basic definitions of formal concept analysis used in this paper. A formal context is a triplet (U, A, I) , where U is a non-empty finite set of objects, A is a nonempty finite set of attributes, and I is a relation between U and A . In the formal context (U, A, I) , for a pair of elements $x \in U$ and $a \in A$, if $(x, a) \in I$, we write xIa . A set of attributes with an object $x \in U$ and a set of objects with an attribute $a \in A$ can be represented as ([1, 5]):

$$x^* = \{a \in A \mid xIa\}; \quad a^* = \{x \in U \mid xIa\}.$$

Extending x to a subset $X \subseteq U$ and a to a subset $B \subseteq A$:

$$X^* = \{a \in A \mid \forall x \in X, xIa\}; \quad B^* = \{x \in U \mid \forall b \in B, xIb\}.$$

A pair (X, B) of two sets $X \subseteq U$ and $B \subseteq A$ is called a *formal concept* of the context (U, A, I) if $X = B^*$ and $B = X^*$ (see [1, 5]). X and B are called the *extent* and the *intent* of the concept, respectively.

Let U be an initial universe set (simply, universe set) and E be a collection of all possible parameters with respect to U , where parameters are the characteristics or properties of objects in U . We will call E the *set of parameters* with respect to U . Let $P(U)$ denote the power set and $A \subseteq E$.

A pair (F, A) is called a *soft set* [12] over U if F is a set-valued mapping of A into the set of all subsets of the set U , i.e.,

$$F : A \rightarrow P(U).$$

In other words, the soft set is a parameterized family of subsets of the set U . Every set $F(e)$, for $e \in A \subseteq E$, from this family may be considered as the set of e -elements of the soft set (F, A) , or as the set of e -approximate elements of the soft set.

We call a soft set (F, A) is *pure* if $\cap_{a \in A} F(a) = \emptyset$ and $F(a) \neq \emptyset$ for every $a \in A$. From now on, we assume that all soft sets are *pure*.

Let $U = \{x_1, x_2, \dots, x_n\}$ be a non-empty finite set of *objects*, $A = \{a_1, a_2, \dots, a_m\}$ a non-empty finite set of *attributes*, and $F : A \rightarrow P(U)$ a soft set. Then the triple (U, A, F) is

called a *soft context* [15].

Let (U, A, F) be a soft context. Then for a soft set (F, A) ,

1) $\mathbf{F}^+ : P(A) \rightarrow P(U)$ is a mapping defined as $\mathbf{F}^+(B) = \cap_{b \in B} F(b)$;

2) $\mathbf{F}^- : P(U) \rightarrow P(A)$ is a mapping defined as $\mathbf{F}^-(X) = \{a \in A : X \subseteq F(a)\}$.

We will denote $\mathbf{Im}(\mathbf{F}^+) = \{\mathbf{F}^+(B) \mid B \in P(A)\}$ and $\mathbf{Im}(\mathbf{F}^-) = \{\mathbf{F}^-(X) \mid X \in P(U)\}$.

Simply, we denote $\mathbf{F}^+(\{a\}) = \mathbf{F}^+(a)$ for each $a \in A$, and $\mathbf{F}^-(\{x\}) = \mathbf{F}^-(x)$ for each $x \in U$.

Theorem 2.1 ([15]). Let (U, A, F) be a soft context, $X, Y \subseteq U$ and $C, D \subseteq A$. Then we have the following things:

1) If $X \subseteq Y$, then $\mathbf{F}^-(Y) \subseteq \mathbf{F}^-(X)$; if $C \subseteq D$, then $\mathbf{F}^+(D) \subseteq \mathbf{F}^+(C)$;

2) $X \subseteq \mathbf{F}^+\mathbf{F}^-(X)$; $C \subseteq \mathbf{F}^-\mathbf{F}^+(C)$;

3) $\mathbf{F}^-(X \cup Y) = \mathbf{F}^-(X) \cap \mathbf{F}^-(Y)$, $\mathbf{F}^+(C \cup D) = \mathbf{F}^+(C) \cap \mathbf{F}^+(D)$;

4) $\mathbf{F}^-(X) = \mathbf{F}^-\mathbf{F}^+\mathbf{F}^-(X)$, $\mathbf{F}^+(C) = \mathbf{F}^+\mathbf{F}^-\mathbf{F}^+(C)$;

5) $\mathbf{F}^-(X) \cup \mathbf{F}^-(Y) \subseteq \mathbf{F}^-(X \cap Y)$, $\mathbf{F}^+(C) \cup \mathbf{F}^+(D) \subseteq \mathbf{F}^+(C \cap D)$.

In a soft context (U, A, F) , let us define two associated operations Ψ_F and Φ_F [15] induced by $\mathbf{F}^+$, $\mathbf{F}^-$ as the following ways:

For each $X \in P(U)$,

$\Psi_F : P(U) \rightarrow P(U)$ is a mapping defined as

$$\Psi_F(X) = \mathbf{F}^+\mathbf{F}^-(X) :$$

In the rest of the paper, Ψ is instead of Ψ_F when there is no ambiguity.

Let (U, A, F) be a soft context and $X \in P(U)$. Then X is called a *soft concept* [15] in (U, A, F) if $\Psi(X) = \mathbf{F}^+\mathbf{F}^-(X) = X$. The set of all soft concepts will be denoted by $s(U, A, F)$.

Theorem 2.2 ([15]). Let (U, A, F) be a soft context. Then

1) $\emptyset, U, \Psi(X)$ are soft concepts.

2) For each $B \subseteq A$, $\mathbf{F}^+(B)$ is a soft concept.

3) For each $a \in A$, $F(a)$ is a soft concept.

4) X is a soft concept if and only if $X = \mathbf{F}^+(B)$ for some $B \in P(A)$.

5) $\mathbf{Im}(\mathbf{F}^+) = s(U, A, F)$.

3. Independence, Dependence of Attributes

For a soft context (U, A, F) , we introduce and study the notions of dependent and independent in A . We will show

that every formal concept of formal context is represented by independent elements of the associated soft context.

From now on, we will denote $|X|$ the cardinal number of any set X .

Definition 3.1. Let (U, A, F) be a soft context. Put $G_a(A) = \{g \in A \mid F(a) \subsetneq F(g)\} \subseteq A$ (simply, G_a). Then for $d \in A$, d is said to be *dependent* on A if there exists $G_d \neq \emptyset$ satisfying $F(d) = \mathbf{F}^+(G_d) = \cap_{a \in G_d} F(a)$.

Otherwise, d is said to be *independent* on A .

We denote: $A_D = \{a \in A \mid a \text{ is dependent on } A\};$
 $A_I = \{a \in A \mid a \text{ is independent on } A\}.$

Example 3.2. Let $U = \{1, 2, 3, 4, 5\}$ and $A = \{a, b, c, d, e, f\}$. Let us consider a soft set $F : A \rightarrow P(U)$ defined by

$$F(a) = F(d) = \{1, 2, 4\}; \quad F(b) = \{2, 4, 5\};$$

$$F(c) = \{2, 4\}; \quad F(e) = \{1, 3\}; \quad F(f) = \{1, 3, 5\}.$$

Then a is independent on A since $G_a = \emptyset$, and also e is independent since $G_e = \{f\} \neq \emptyset$ but $\mathbf{F}^+(G_e) = \cap_{g \in G_e} F(g) = F(f) \neq F(e)$. In case of $c \in A$, it is dependent since $G_c = \{a, b, d\} \neq \emptyset$ and $\mathbf{F}^+(G_c) = \cap_{g \in G_c} F(g) = F(c)$.

Consequently, we have

$$A_I = \{a, b, d, e, f\}; \quad A_D = \{c\}.$$

Then we easily obtain the following facts:

Theorem 3.3. Let (U, A, F) be a soft context. Then

- 1) $A_D \cap A_I = \emptyset$; $A_D \cup A_I = A$.
- 2) a is independent if and only if either $G_a = \emptyset$ or if $G_a \neq \emptyset$, then $\mathbf{F}^+(G_a) = \cap_{g \in G_a} F(g) \neq F(a)$.

Definition 3.4. Let (U, A, F) be a soft context. For $a \in A$, we say that the element a is *generated* by finitely many elements $b_1, b_2, \dots, b_n \in A$ if $F(a) = \cap_{b \in B} F(b)$ for some $B = \{b_1, b_2, \dots, b_n\} \subseteq A$, and $b \in B$ is called *generator* for a .

Example 3.5. In Example 3.2, for $c \in A$, a, b and d are generators for c .

Lemma 3.6. Let (U, A, F) be a soft context. For $d \in A_D$, $G_d = \{g \in A \mid F(d) \subsetneq F(g)\}$ is the maximal set of generators.

Theorem 3.7. Let (U, A, F) be a soft context. Then every dependent element of A is generated by finitely many independent

elements of A , that is, for each $d \in A_D$, there exists $B \subseteq A_I$ such that $\mathbf{F}^+(B) = \cap_{b \in B} F(b) = F(d)$.

Proof. Suppose that there is a dependent element d of A such that it can not be generated only by independent elements of A .

Put $\mathcal{D} = \{g \in A_D \mid g \text{ is not generated only by independent elements of } A\}$. Then by hypothesis, $\mathcal{D}$ is not empty. Furthermore, $\mathcal{D} \cap G_d \neq \emptyset$.

For the proof, assume that $|\mathcal{D}| = n < |A_D| < |A|$.

First, pick up one element in $\mathcal{D}$, say d_1 . Then since $d_1 \in \mathcal{D} \subset A_D$, $G_{d_1} = \{a \in A \mid F(d_1) \subsetneq F(a)\}$ is not empty set and $G_{d_1} \cap \mathcal{D} \neq \emptyset$. So, we can pick up $d_2 \in G_{d_1} \cap \mathcal{D}$. Then obviously, $F(d_1) \subsetneq F(d_2)$.

Second, for $d_2 \in \mathcal{D} \subset A_D$, let us consider $G_{d_2} = \{a \in A \mid F(d_2) \subsetneq F(a)\}$. Then since $d_2 \in \mathcal{D}$ and G_{d_2} is the maximal set of generators for d_2 , $G_{d_2} \cap \mathcal{D} \neq \emptyset$. Now, pick up $d_3 \in G_{d_2} \cap \mathcal{D}$. Then obviously, $F(d_2) \subsetneq F(d_3)$ and so, $F(d_1) \subsetneq F(d_2) \subsetneq F(d_3)$.

Repeating this process, after a finite number $(n - 2)$, we get an element $d_{n-1} \in \mathcal{D}$. Then it satisfies that $F(d_1) \subsetneq \dots \subsetneq F(d_{n-2}) \subsetneq F(d_{n-1})$ and $G_{d_{n-1}} \cap \mathcal{D} \neq \emptyset$.

Finally, since $|\mathcal{D}| = n$, we can pick up the last element $d_n \in G_{d_{n-1}} \cap \mathcal{D}$. Then $F(d_1) \subsetneq F(d_2) \subsetneq \dots \subsetneq F(d_{n-1}) \subsetneq F(d_n)$.

In the last step, since d_n is the last element in $\mathcal{D}$, for $G_{d_n} = \{a \in A \mid F(d_n) \subsetneq F(a)\}$, $G_{d_n} \cap \mathcal{D} = \emptyset$. So, the last element $d_n \in \mathcal{D}$ is generated by finitely many independent elements of A . From this fact and $F(d_1) \subsetneq \dots \subsetneq F(d_{n-2}) \subsetneq F(d_{n-1}) \subsetneq F(d_n)$, we know that $d_{n-1} \in \mathcal{D}$ should be generated by finitely many independent elements of A . Consequently, $\mathcal{D}$ is the empty set. So the proof is completed. $\square$

Theorem 3.8. Let (U, A, F) be a soft context. Then $\cap_{a \in A_I} F(a) = \emptyset$.

Proof. From Theorem 2.1 and Theorem 3.7, it follows $\mathbf{F}^+(A_I) \cap \mathbf{F}^+(A_D) = \mathbf{F}^+(A_I)$ and $\cap_{a \in A_I} F(a) = \mathbf{F}^+(A_I) = \mathbf{F}^+(A_I) \cap \mathbf{F}^+(A_D) = \mathbf{F}^+(A_I \cup A_D) = \mathbf{F}^+(A) = \cap_{a \in A} F(a)$. Since the soft set (F, A) is pure, $\cap_{a \in A_I} F(a) = \emptyset$. $\square$

Theorem 3.9. Let (U, A, F) be a soft context. Put $\mathcal{A}_{\mathcal{I}} = \{F(a) \mid a \in A_I\}$ for $A_I \subseteq A$. Then

$$s(U, A, F) = \{\cap \mathcal{S} \mid \mathcal{S} \subseteq \mathcal{A}_{\mathcal{I}}\}.$$

Proof. 1) Take $\mathcal{S} = \mathcal{A}_{\mathcal{I}}$. Then from the theorem above, $\cap \mathcal{S} = \cap_{a \in A_I} F(a) = \emptyset$. It is obvious that $\mathcal{A}_{\mathcal{I}} \subseteq \mathbf{Im}(\mathbf{F}^+) = s(U, A, F)$. Take $\mathcal{S} = \emptyset \subseteq \mathcal{A}_{\mathcal{I}}$. Then clearly, $\cap \mathcal{S} = U$.

2) Let $X \in s(U, A, F)$. Then, from (4) of Theorem 2.2, there exists $B \subseteq A$ such that $X = \mathbf{F}^+(B)$. Let $C = B \cap A_I$ and $D = B \cap A_D$. From $B = C \cup D$ and $C \cap D = \emptyset$, it follows $X = \mathbf{F}^+(B) = \mathbf{F}^+(C \cup D) = \mathbf{F}^+(C) \cap \mathbf{F}^+(D) = \mathbf{F}^+(C) \cap (\cap_{d \in D} F(d))$. So, from Theorem 3.7, for each $d \in D \subseteq A_D$, there exists $E_d \subseteq A_I$ such that $\mathbf{F}^+(E_d) = \cap_{e \in E_d} F(e) = F(d)$. Put $H = \cup_{d \in D} E_d$. Then $H \subseteq A_I$ and $\mathbf{F}^+(H) = \mathbf{F}^+(\cup_{d \in D} E_d) = \cap_{d \in D} \mathbf{F}^+(E_d) = \cap_{d \in D} F(d)$. It implies $X = \mathbf{F}^+(B) = \mathbf{F}^+(C) \cap (\cap_{d \in D} F(d)) = \mathbf{F}^+(C) \cap \mathbf{F}^+(H) = \mathbf{F}^+(C \cup H)$ for $C \cup H \subseteq A_I$. Finally, put $\mathcal{S} = \{F(s) | s \in C \cup H \subseteq A_I\}$; then $\mathcal{S} \subseteq \mathcal{A}_I$ and $\cap \mathcal{S} = \cap_{s \in C \cup H} F(s) = \mathbf{F}^+(C \cup H) = X$. So the proof is completed. $\square$

For a formal context (U, A, I) , let us define a soft set $F_I : A \rightarrow P(U)$ as follows $F_I(a) = \{x \in U : (x, a) \in I\}$. Then (U, A, F_I) is a soft context. So, every formal context (U, A, I) induces a soft context (U, A, F_I) . In this paper, we call (U, A, F_I) the *associated soft context* induced by a formal context (U, A, I) .

Theorem 3.10. Let (U, A, I) be a formal context. Then for the associated soft context (U, A, F_I) , $s(U, A, F_I) = \{\mathbf{F}^+(B) | B \text{ is any subset of } A_I\}$.

Proof. For the associated soft context (U, A, F_I) , from Theorem 3.9, it is obvious that $X \in \{\mathbf{F}^+(B) | B \text{ is any subset of } A_I\}$ if and only if $X \in \{\cap \mathcal{S} | \mathcal{S} \subseteq \mathcal{A}_I\}$. So the fact is obtained. $\square$

In [1], for a formal context (U, A, I) , The concepts of (U, A, I) are ordered by

$$(X_1, B_1) \leq (X_2, B_2) \Leftrightarrow X_1 \subseteq X_2 (\Leftrightarrow B_1 \supseteq B_2),$$

where $(X_1, B_1), (X_2, B_2)$ are two concepts.

(X_1, B_1) is called a *sub-concept* of (X_2, B_2) , and (X_2, B_2) is called a *super-concept* of (X_1, B_1) . The ordered set of all concepts in (U, A, I) is denoted by $L(U, A, I)$ and called the *concept lattice* of (U, A, I) , where the infimum and supremum are defined by:

$$\begin{aligned} (X_1, B_1) \wedge (X_2, B_2) &= (X_1 \cap X_2, (B_1 \cup B_2)^{**}), \\ (X_1, B_1) \vee (X_2, B_2) &= ((X_1 \cup X_2)^{**}, B_1 \cap B_2). \end{aligned}$$

Theorem 3.11 ([15]). Let (U, A, I) be a formal context. Then

$$L(U, A, I) = \{(X, \mathbf{F}_I^-(X)) | X \text{ is any element of } s(U, A, F_I)\}.$$

By Theorem 3.10 and Theorem 3.11, the following theorem is obtained:

Theorem 3.12. Let (U, A, I) be a formal context. Then

$$L(U, A, I) = \{(F^+(B), F^- F^+(B)) | B \text{ is any subset of } A_I \text{ in } (U, A, F_I)\}.$$

4. Conclusions

We introduced the notion of independent and dependent attributes in a given soft context. Then we showed that every dependent attribute is generated by some independent attributes in a given soft context. In the next research, we will study special properties of the independent attributes, and characterizations for soft concepts and soft concept lattice by using a nonempty finite set of attributes. Furthermore, the results of this paper will be applied to the reduction of formal concepts and the research of the formal concept analysis.

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