Categories of Alexandrov L -Fuzzy Pre-uniformities, L -Approximation Operators and Alexandrov L -Fuzzy Topologies

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Abstract

This paper investigates relationships among Alexandrov L -fuzzy pre-uniformities, Alexandrov L -fuzzy topologies, L -lower approximation operators and L -upper approximation operators. It also presents an adjointness between the category of Alexandrov L -fuzzy pre-uniform spaces and reflexive L -fuzzy relations. In addition, it shows that the category of Alexandrov L -fuzzy pre-uniform spaces and the category of L -lower (respectively, upper) approximation spaces are isomorphic.

Keywords: Complete residuated lattice, Alexandrov fuzzy topology, Lower approximation operator, Upper approximation operator, Alexandrov fuzzy pre-uniformity

1. Introduction

Pawlak [1, 2] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. This theory was extended and applied in many directions [3–15]. An interesting and natural research topic in rough set theory is the study of algebraic structure and topological structure.

Ward and Dilworth [16] introduced a complete residuated lattice which is an important mathematical tool as algebraic structures for many valued logics [3–8, 10, 17–25]. For an extension of classical rough sets, many researchers [3, 4, 6, 21, 24] developed L -lower and L -upper approximation operators in complete residuated lattices. By using this concepts, information systems and decision rules were investigated in complete residuated lattices [6, 7, 17, 18, 20]. Cimoka and Sostak [18] and Ramadan et al. [22, 23] investigated the relationships between L -fuzzy quasi-uniformities and L -fuzzy topologies in complete residuated lattices.

Ma [10] investigated the topological and lattice structures of L -fuzzy rough sets determined by lower and upper sets. Kim and Kim [5] studied the relations between L -fuzzy upper and lower approximation spaces and L -fuzzy quasi-uniform spaces in a strictly two-sided, commutative quantale. Kim and his colleagues [3, 4, 10] investigated the properties of L -lower and L -upper approximate operators and Alexandrov L -fuzzy topologies in complete residuated lattices.

In this paper, we investigate relationships among Alexandrov L -fuzzy pre-uniformities, Alexandrov L -fuzzy topologies, L -lower approximation operators and L -upper approximation operators. We obtain an adjointness between the category of Alexandrov L -fuzzy pre-uniform

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spaces and reflexive L -fuzzy relations. In addition, we show that the category of Alexandrov L -fuzzy pre-uniform spaces and the category of L -lower (resp. upper) approximation spaces are isomorphic.

Definition 1.1 ([6, 7, 17]). An algebra $(L, \wedge, \vee, \odot, \rightarrow, \perp, \top)$ is called a *complete residuated lattice* if it satisfies the following conditions:

- (L1) $(L, \leq, \vee, \wedge, \perp, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\perp$;
- (L2) $(L, \odot, \top)$ is a commutative monoid;
- (L3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we always assume that $(L, \leq, \odot, \rightarrow, *)$ is complete residuated lattice with an order reversing involution $*$ defined by

$$x^* = x \rightarrow \perp, \quad (x^*)^* = x.$$

2. Preliminaries

For $\alpha \in L, f \in L^X$, we denote $(\alpha \rightarrow f), (\alpha \odot f), \alpha_X \in L^X$ as $(\alpha \rightarrow f)(x) = \alpha \rightarrow f(x), (\alpha \odot f)(x) = \alpha \odot f(x), \alpha_X(x) = \alpha, \top_x \in L^X, \top_{(x,y)} \in L^{X \times X}$,

$$\begin{aligned} \top_x(y) &= \begin{cases} \top, & \text{if } y = x, \\ \perp, & \text{otherwise,} \end{cases} \\ \top_{(x,y)}(z, w) &= \begin{cases} \top, & \text{if } (x, y) = (z, w), \\ \perp, & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 2.1 ([6, 7, 17]). For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

- (1) $\top \rightarrow x = x, \perp \odot x = \perp$,
- (2) If $y \leq z$, then $x \odot y \leq x \odot z, x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$,
- (3) $x \leq y$ iff $x \rightarrow y = \top$,
- (4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*, (\bigvee_i y_i)^* = \bigwedge_i y_i^*$,
- (5) $x \rightarrow (\bigwedge_i y_i) = \bigwedge_i (x \rightarrow y_i)$,
- (6) $(\bigvee_i x_i) \rightarrow y = \bigwedge_i (x_i \rightarrow y)$,
- (7) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$,
- (8) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
- (9) $x \odot y = (x \rightarrow y^*)^*, x \rightarrow y = y^* \rightarrow x^*$,
- (10) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \odot z) \rightarrow (y \odot w)$ and $(x \rightarrow y) \rightarrow (x \rightarrow z) \geq y \rightarrow z$,
- (11) $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$ and $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$,
- (12) $\bigvee_{i \in \Gamma} x_i \rightarrow \bigvee_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \rightarrow y_i)$ and $\bigwedge_{i \in \Gamma} x_i \rightarrow \bigwedge_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \rightarrow y_i)$.

Definition 2.2 ([6, 7, 17]). Let X be a set. A map $R : X \times X \rightarrow L$ is called an *L -fuzzy preorder* if it satisfies the following conditions:

- (E1) reflexive if $R(x, x) = \top$ for all $x \in X$,
- (E2) transitive if $R(x, y) \odot R(y, z) \leq R(x, z)$, for all $x, y, z \in X$.

Lemma 2.3 ([6, 7, 17]). For a given set X , define a binary map $S : L^X \times L^X \rightarrow L$ by

$$S(f, g) = \bigwedge_{x \in X} (f(x) \rightarrow g(x)).$$

Then, for each $f, g, h, k \in L^X$, and $\alpha \in L$, the following properties hold.

- (1) S is an L -fuzzy preorder on L^X .
- (2) $f \leq g$ iff $S(f, g) = \top$.
- (3) If $f \leq g$, then $S(h, f) \leq S(h, g)$ and $S(f, h) \geq S(g, h)$.
- (4) $S(f, g) \odot S(k, h) \leq S(f \odot k, g \odot h)$.
- (5) $S(g, h) \leq S(f, g) \rightarrow S(f, h)$.
- (6) $S(f, h) = \bigvee_{g \in L^X} (S(f, g) \odot S(g, h))$.

Definition 2.4 ([3, 4, 20]). A map $\mathcal{T} : L^X \rightarrow L$ is called an *Alexandrov L -fuzzy topology* on X if it satisfies the following conditions:

- (AT1) $\mathcal{T}(\perp_X) = \mathcal{T}(\top_X) = \top$,
- (AT2) $\mathcal{T}(\bigwedge_i f_i) \geq \bigwedge_i \mathcal{T}(f_i)$ and $\mathcal{T}(\bigvee_i f_i) \geq \bigwedge_i \mathcal{T}(f_i)$, for all $\{f_i\}_{i \in \Gamma} \subseteq L^X$.
- (AT3) $\mathcal{T}(\alpha \odot f) \geq \mathcal{T}(f)$ and $\mathcal{T}(\alpha \rightarrow f) \geq \mathcal{T}(f)$, for all $\alpha \in L, f \in L^X$.

The pair $(X, \mathcal{T})$ is called an *Alexandrov L -fuzzy topological space*.

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two Alexandrov L -fuzzy topological spaces. A map $\varphi : X \rightarrow Y$ is called *L -fuzzy continuous* if for each $f \in L^Y$,

$$\mathcal{T}_2(f) \leq \mathcal{T}_1(\varphi^{\leftarrow}(f)).$$

Definition 2.5 ([10]). A map $\mathcal{U} : L^{X \times X} \rightarrow L$ is called an *Alexandrov L -fuzzy pre-uniformity* on X iff the following conditions hold.

- (AU1) There exists $u \in L^{X \times X}$ such that $\mathcal{U}(u) = \top$.
- (AU2) If $v \leq u$, then $\mathcal{U}(v) \leq \mathcal{U}(u)$.
- (AU3) For every $u_i \in L^{X \times X}, \mathcal{U}(\bigwedge_{i \in \Gamma} u_i) = \bigwedge_i \mathcal{U}(u_i)$.
- (AU4) $\mathcal{U}(u) \leq \bigwedge_{x \in X} u(x, x)$.
- (AU5) $\mathcal{U}(\alpha \rightarrow u) = \alpha \rightarrow \mathcal{U}(u)$ for each $\alpha \in L$.

The pair $(X, \mathcal{U})$ is called an *Alexandrov L -fuzzy pre-uniform space*.

An Alexandrov L -fuzzy pre-uniformity on X is called an *Alexandrov L -fuzzy quasi-uniformity* if

(AQ) $\mathcal{U}(u) \leq \bigvee \{ \mathcal{U}(v) \odot \mathcal{U}(w) \mid v \circ w \leq u \}$, where

$$v \circ w(x, z) = \bigvee_{y \in X} u(x, y) \odot v(y, z).$$

Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be an Alexandrov L -fuzzy quasi-uniform spaces, and $\varphi : X \rightarrow Y$ be a map. Then φ is said to be *L -fuzzy uniformly continuous* if $\mathcal{V}(v) \leq \mathcal{U}((\varphi \times \varphi)^{\leftarrow}(v))$, for every $v \in L^{Y \times Y}$.

Remark 2.6. Let $(X, \mathcal{U})$ be an L -fuzzy pre-uniform space. By (AU1) and (AU2), we have $\mathcal{U}(\top_{X \times X}) = \top$, because $u \leq \top_{X \times X}$ for all $u \in L^{X \times X}$.

Definition 2.7 ([3]). A map $\mathcal{J} : L^X \rightarrow L^X$ is called an *L -lower approximation operator on X* if

- (J1) $\mathcal{J}(\top_X) = \top_X$,
- (J2) $\mathcal{J}(f) \leq f$ for all $f \in L^X$,
- (J3) $\mathcal{J}(\bigwedge_{i \in \Gamma} f_i) = \bigwedge_{i \in \Gamma} \mathcal{J}(f_i)$ for all $f_i \in L^X$, and
- (J4) $\mathcal{J}(\alpha \rightarrow f) = \alpha \rightarrow \mathcal{J}(f)$.

An L -lower approximation operator is said to be *topological* if (T) $\mathcal{J}(\mathcal{J}(f)) = \mathcal{J}(f)$ for all $f \in L^X$.

Let $(X, \mathcal{J}_X)$ and $(Y, \mathcal{J}_Y)$ be L -lower approximation spaces. $\varphi : (X, \mathcal{J}_X) \rightarrow (Y, \mathcal{J}_Y)$ is called an *L -lower approximation map* if, for each $g \in L^Y$,

$$\varphi^{\leftarrow}(\mathcal{J}_Y(g)) \leq \mathcal{J}_X(\varphi^{\leftarrow}(g)).$$

Definition 2.8 ([3]). A map $\mathcal{H} : L^X \rightarrow L^X$ is called an *L -upper approximation operator on X* if

- (H1) $\mathcal{H}(\perp_X) = \perp_X$,
- (H2) $\mathcal{H}(f) \geq f$ for all $f \in L^X$,
- (H3) $\mathcal{H}(\bigvee_{i \in \Gamma} f_i) = \bigvee_{i \in \Gamma} \mathcal{H}(f_i)$ for all $f_i \in L^X$, and
- (H4) $\mathcal{H}(\alpha \odot f) = \alpha \odot \mathcal{H}(f)$.

An L -upper approximation operator is said to be *topological* if (T) $\mathcal{H}(\mathcal{H}(f)) = \mathcal{H}(f)$, for all $f \in L^X$.

Let $\mathcal{H}$ and $\mathcal{J}$ be an L -upper and L -lower approximation on X , respectively. The pair $(\mathcal{J}(f), \mathcal{H}(f))$ is called a *fuzzy rough set* for f .

Let $(X, \mathcal{H}_X)$ and $(Y, \mathcal{H}_Y)$ be L -upper approximation spaces. Then $\varphi : (X, \mathcal{H}_X) \rightarrow (Y, \mathcal{H}_Y)$ is called an *L -upper approximation map* if, for each $g \in L^Y$,

$$\varphi^{\leftarrow}(\mathcal{H}_Y(g)) \geq \mathcal{H}_X(\varphi^{\leftarrow}(g)).$$

3. Categories of Alexandrov L -Fuzzy Pre-uniformities, L -Approximation Operators and Alexandrov L -Fuzzy Topologies

Theorem 3.1. Let $\mathcal{U}$ be an Alexandrov L -fuzzy pre-uniformity on X . For each $u \in L^{X \times X}$ with $u^{-1}(x, y) = u(y, x)$, define a map $\mathcal{U}^s : L^{X \times X} \rightarrow L$ by

$$\mathcal{U}^s(u) = \mathcal{U}(u^{-1}).$$

Then the following properties hold.

- (1) $\mathcal{U}^s$ is an Alexandrov L -fuzzy pre-uniformity on X .
- (2) If $\mathcal{U}$ is an Alexandrov L -fuzzy quasi-uniformity on X , then $\mathcal{U}^s$ is an Alexandrov L -fuzzy quasi-uniformity on X .

Proof. (1) (AU1), (AU2) and (AU3) can be easily proved. (AU4) For all $u \in L^{X \times X}$, we have

$$\mathcal{U}^s(u) = \mathcal{U}(u^{-1}) \leq \bigwedge_{x \in X} u^{-1}(x, x) = \bigwedge_{x \in X} u(x, x)$$

(AU5) For all $\alpha \in L$ and $u \in L^{X \times X}$,

$$\mathcal{U}^s(\alpha \rightarrow u) = \mathcal{U}(\alpha \rightarrow u^{-1}) = \alpha \rightarrow \mathcal{U}(u^{-1}) = \alpha \rightarrow \mathcal{U}^s(u).$$

Hence $\mathcal{U}^s$ is an Alexandrov L -fuzzy pre-uniformity on X .

(2) (AQ) For all $u \in L^{X \times X}$,

$$\begin{aligned} \mathcal{U}^s(u) &= \mathcal{U}(u^{-1}) \\ &\leq \bigvee \{ \mathcal{U}(v^{-1}) \odot \mathcal{U}(w^{-1}) \mid v^{-1} \circ w^{-1} \leq u^{-1} \} \\ &= \bigvee \{ \mathcal{U}^s(v) \odot \mathcal{U}^s(w) \mid v \circ w \leq u \}. \end{aligned}$$

□

Theorem 3.2. Let $R \in L^{X \times X}$ be a reflexive L -fuzzy relation. Define a map $\mathcal{U}_R : L^{X \times X} \rightarrow L$ as

$$\mathcal{U}_R(u) = \bigwedge_{x, y \in X} (R(x, y) \rightarrow u(x, y)).$$

Then the following properties hold.

- (1) $\mathcal{U}_R$ is an Alexandrov L -fuzzy pre-uniformity on X .
- (2) $\mathcal{U}_R^s$ is an Alexandrov L -fuzzy pre-uniformity on X with $\mathcal{U}_R^s = \mathcal{U}_{R^{-1}}$.
- (3) If R is an L -fuzzy preorder and there exists $x \neq y \in X$ such that $R(x, y) = \perp$, then $\mathcal{U}_R$ is an Alexandrov L -fuzzy quasi-uniformity on X .

Proof. (AU1) There exists $\top_{X \times X} \in L^{X \times X}$ such that $\mathcal{U}_R(\top_{X \times X}) = \top$.

(AU2), (AU3) and (AU5) can be easily proved. (AU4)

$$\begin{aligned}\mathcal{U}(u) &= \bigwedge_{x,y \in X} (R(x,y) \rightarrow u(x,y)) \\ &= \bigwedge_{x,y \in X} (R(x,y) \rightarrow u(x,y)) \\ &\leq \bigwedge_{x \in X} (R(x,x) \rightarrow u(x,x)) = \bigwedge_{x \in X} u(x,x).\end{aligned}$$

Hence $\mathcal{U}$ is an Alexandrov L -fuzzy pre-uniformity on X .

(2) We obtain $\mathcal{U}_R^s : L^{X \times X} \rightarrow L$ as

$$\begin{aligned}\mathcal{U}_R^s(u) &= \mathcal{U}(u^{-1}) \\ &= \bigwedge_{x,y \in X} (R(x,y) \rightarrow u^{-1}(x,y)) \\ &= \bigwedge_{x,y \in X} (R^{-1}(x,y) \rightarrow u(x,y)) = \mathcal{U}_{R^{-1}}(u).\end{aligned}$$

(3) For $v, w \in L^{X \times X}$,

$$\begin{aligned}\mathcal{U}(v) \odot \mathcal{U}(w) &= \bigwedge_{x,y \in X} (R(x,y) \rightarrow v(x,y)) \odot \bigwedge_{z,w \in X} (R(z,w) \rightarrow w(z,w)) \\ &\leq \bigwedge_{x,y,z \in X} (R(x,y) \rightarrow v(x,y)) \odot (R(z,w) \rightarrow w(z,w)) \\ &\leq \bigwedge_{x,y,z \in X} (R(x,y) \odot R(y,z) \rightarrow v(x,y) \odot w(y,z)) \\ &\leq \bigwedge_{x,z \in X} \left(\bigvee_{y \in X} (R(x,y) \odot R(y,z)) \right) \\ &\quad \rightarrow \bigvee_{y \in X} (v(x,y) \odot w(y,z)) \\ &\leq \bigwedge_{x,z \in X} (R(x,z) \rightarrow u(x,y)) = \mathcal{U}(u).\end{aligned}$$

Hence $\mathcal{U}(u) \geq \bigvee \{\mathcal{U}(v) \odot \mathcal{U}(w) \mid v \circ w \leq u\}$.

Since $\triangle_{X \times X} \circ u = u$ and

$$\begin{aligned}\mathcal{U}_R(\triangle_{X \times X}) &= \bigwedge_{x,y \in X} (R(x,y) \rightarrow \triangle_{X \times X}(x,y)) \\ &= \bigvee_{x \neq y \in X} R^*(x,y) = \top,\end{aligned}$$

we have $\bigvee \{\mathcal{U}(v) \odot \mathcal{U}(w) \mid v \circ w \leq u\} \geq \mathcal{U}_R(\triangle_{X \times X}) \odot \mathcal{U}_R(u) = \mathcal{U}_R(u)$. $\square$

Example 3.3. Let $([0, 1], \odot, \rightarrow, *, 0, 1)$ be a complete residu-

ated lattice (ref. [17, 19, 20, 25]) as

$$x \odot y = \max\{0, x + y - 1\}, \quad x \rightarrow y = \min\{1 - x + y, 1\}$$

$$x^* = 1 - x.$$

Let $X = \{x, y, z\}$ and an L -fuzzy preorder $R \in [0, 1]^{X \times X}$ as

$$R = \begin{pmatrix} 1 & 0.4 & 0 \\ 0.8 & 1 & 0.5 \\ 0.5 & 0.6 & 1 \end{pmatrix}.$$

From Theorem 3.2(3), we obtain Alexandrov L -fuzzy quasi-uniformities $\mathcal{U}_R, \mathcal{U}_R^s : L^{X \times X} \rightarrow L$ as

$$\begin{aligned}\mathcal{U}_R(u) &= \bigwedge_{x,y \in X} (R(x,y) \rightarrow u(x,y)), \\ \mathcal{U}_R^s(u) &= \bigwedge_{x,y \in X} (R^{-1}(x,y) \rightarrow u(x,y)).\end{aligned}$$

Theorem 3.4. Let $(X, \mathcal{U})$ be an Alexandrov L -fuzzy preuniform space. Define a map $R_{\mathcal{U}} : X \times X \rightarrow L$ by

$$R_{\mathcal{U}}(x, y) = \bigwedge_{u \in L^{X \times X}} (\mathcal{U}(u) \rightarrow u(x, y)).$$

Then the following properties hold.

(1) $R_{\mathcal{U}}$ is a reflexive L -fuzzy relation such that $R_{\mathcal{U}}(x, y) = \mathcal{U}^*(\top_{(x,y)})$.

(2) If $(X, \mathcal{U})$ be an Alexandrov L -fuzzy quasi-uniform space, then $R_{\mathcal{U}}$ is an L -fuzzy preorder.

(3) $\mathcal{U} \leq \mathcal{U}_{R_{\mathcal{U}}}$.

(4) If $R \in L^{X \times X}$ is reflexive, then $R = R_{\mathcal{U}_R}$.

Proof. (1) For any $x \in X$, we have $\mathcal{U}(u) \leq u(x, x)$ from (AU2). Then

$$R_{\mathcal{U}}(x, x) = \bigwedge_{u \in L^{X \times X}} (\mathcal{U}(u) \rightarrow u(x, x)) = \top.$$

Thus $R_{\mathcal{U}}$ is an reflexive L -fuzzy relation. For each

$u = \bigwedge_{x,y \in X} (u^*(x, y) \rightarrow \top_{(x,y)})$, we have

$$\begin{aligned}R_{\mathcal{U}}(x, y) &= \bigwedge_{u \in L^{X \times X}} (\mathcal{U}(u) \rightarrow u(x, y)) \\ &= \bigwedge_{u \in L^{X \times X}} (\mathcal{U}(\bigwedge_{x,y \in X} (u^*(x, y) \rightarrow \top_{(x,y)})) \rightarrow u(x, y)) \\ &= \bigwedge_{u \in L^{X \times X}} (\bigwedge_{x,y \in X} (u^*(x, y) \rightarrow \mathcal{U}(\top_{(x,y)})) \rightarrow u(x, y))\end{aligned}$$

$$\begin{aligned}
 &= \bigwedge_{u \in L^{X \times X}} \left(\bigwedge_{x, y \in X} (\mathcal{U}^*(\top_{(x,y)}^*) \rightarrow u(x, y)) \rightarrow u(x, y) \right) \\
 &\geq \mathcal{U}^*(\top_{(x,y)}^*).
 \end{aligned}$$

Also, we have

$$\begin{aligned}
 R_{\mathcal{U}}(x, y) &= \bigwedge_{u \in L^{X \times X}} (\mathcal{U}(u) \rightarrow u(x, y)) \\
 &\leq \mathcal{U}^*(\top_{(x,y)}^*) \rightarrow \top_{(x,y)}^*(x, y) = \mathcal{U}^*(\top_{(x,y)}^*).
 \end{aligned}$$

Hence $R_{\mathcal{U}}(x, y) = \mathcal{U}^*(\top_{(x,y)}^*)$.

(2) For each $x, y, z \in X$ and $v \circ w \leq u$,

$$\begin{aligned}
 &R_{\mathcal{U}}(x, y) \odot R_{\mathcal{U}}(y, z) \\
 &= \bigwedge_{v \in L^{X \times X}} (\mathcal{U}(v) \rightarrow v(x, y)) \odot \bigwedge_{w \in L^{X \times X}} (\mathcal{U}(w) \rightarrow w(y, z)) \\
 &\leq \bigwedge_{v, w \in L^{X \times X}} (\mathcal{U}(v) \odot \mathcal{U}(w) \rightarrow v(x, y) \odot w(y, z)) \\
 &\leq \bigwedge_{u \in L^{X \times X}} (\mathcal{U}(u) \rightarrow u(x, z)) = R_{\mathcal{U}}(x, z) \text{ (by AQ)}.
 \end{aligned}$$

(3) For each $u \in L^{X \times X}$,

$$\begin{aligned}
 \mathcal{U}_{R_{\mathcal{U}}}(u) &= \bigwedge_{x, y \in X} (R_{\mathcal{U}}(x, y) \rightarrow u(x, y)) \\
 &= \bigwedge_{x, y \in X} \left(\bigwedge_{v \in L^{X \times X}} (\mathcal{U}(v) \rightarrow v(x, y)) \rightarrow u(x, y) \right) \\
 &\geq \bigwedge_{x, y \in X} ((\mathcal{U}(u) \rightarrow u(x, y)) \rightarrow u(x, y)) \\
 &\geq \mathcal{U}(u).
 \end{aligned}$$

(4) For each $x, y \in X$, we have

$$\begin{aligned}
 &R_{\mathcal{U}_R}(x, y) \\
 &= \bigwedge_{u \in L^{X \times X}} (\mathcal{U}_R(u) \rightarrow u(x, y)) \\
 &= \bigwedge_{u \in L^{X \times X}} \left(\bigwedge_{x, y \in X} (R(x, y) \rightarrow u(x, y)) \rightarrow u(x, y) \right) \\
 &\geq \bigwedge_{u \in L^{X \times X}} ((R(x, y) \rightarrow u(x, y)) \rightarrow u(x, y)) \\
 &\geq R(x, y).
 \end{aligned}$$

Also, we have

$$\begin{aligned}
 R_{\mathcal{U}_R}(x, y) &\leq (\mathcal{U}_R(\top_{(x,y)}^*) \rightarrow \top_{(x,y)}^*(x, y)) \\
 &= (R(x, y) \rightarrow \top_{(x,y)}^*(x, y)) \rightarrow \top_{(x,y)}^*(x, y) \\
 &= R(x, y).
 \end{aligned}$$

Hence, $R = R_{\mathcal{U}_R}$.

Theorem 3.5. Let $(X, \mathcal{U})$ be an Alexandrov L -pre-uniform space. Define two maps $\mathcal{J}_{\mathcal{U}}, \mathcal{H}_{\mathcal{U}} : L^X \rightarrow L^X$ by

$$\begin{aligned}
 \mathcal{J}_{\mathcal{U}}(f)(x) &= \bigwedge_{y \in X} (\mathcal{U}^*(\top_{(x,y)}^*) \rightarrow f(y)), \\
 \mathcal{H}_{\mathcal{U}}(f)(x) &= \bigvee_{y \in X} (\mathcal{U}^*(\top_{(x,y)}^*) \odot f(y)).
 \end{aligned}$$

Then the following properties hold.

(1) $\mathcal{J}_{\mathcal{U}}$ is an L -lower approximation operator such that

$$\mathcal{J}_{\mathcal{U}}(\top_y^*)(x) = \mathcal{U}(\top_{(x,y)}^*).$$

(2) $\mathcal{H}_{\mathcal{U}}$ is an L -upper approximation operator such that

$$\mathcal{H}_{\mathcal{U}}(\top_y^*)(x) = \mathcal{U}^*(\top_{(x,y)}^*).$$

(3) If $(X, \mathcal{U})$ be an Alexandrov L -fuzzy quasi-uniform space, then $\mathcal{J}_{\mathcal{U}}$ is a topological L -lower approximation operator.

(4) If $(X, \mathcal{U})$ be an Alexandrov L -fuzzy quasi-uniform space, then $\mathcal{H}_{\mathcal{U}}$ is a topological L -upper approximation operator.

Proof. (1) Since $\mathcal{U}(\top_{(x,x)}^*) \leq \top_{(x,x)}^*(x, x) = \perp$ from (AU4), we have (J2)

$$\mathcal{J}_{\mathcal{U}}(f)(x) \leq \mathcal{U}^*(\top_{(x,x)}^*) \rightarrow f(x) = f(x).$$

(J1), (J3), (J4) and (2) are easily proved.

(3) By Theorem 3.4(2), $\bigvee_{y \in X} (\mathcal{U}^*(\top_{(x,y)}^*) \odot \mathcal{U}^*(\top_{(y,z)}^*)) = \bigvee_{y \in X} (R_{\mathcal{U}}(x, y) \odot R_{\mathcal{U}}(y, z)) = R_{\mathcal{U}}(x, z) = \mathcal{U}^*(\top_{(x,z)}^*)$. For each $x \in X, f \in L^X$,

$$\begin{aligned}
 &\mathcal{J}_{\mathcal{U}}(\mathcal{J}_{\mathcal{U}}(f))(x) \\
 &= \bigwedge_{y \in X} (\mathcal{U}^*(\top_{(x,y)}^*) \rightarrow \mathcal{J}_{\mathcal{U}}(f)(y)) \\
 &= \bigwedge_{y \in X} (\mathcal{U}^*(\top_{(x,y)}^*) \rightarrow \bigwedge_{z \in X} (\mathcal{U}^*(\top_{(y,z)}^*) \rightarrow f(z))) \\
 &= \bigwedge_{z \in X} \left(\bigvee_{y \in X} (\mathcal{U}^*(\top_{(x,y)}^*) \odot \mathcal{U}^*(\top_{(y,z)}^*)) \rightarrow f(z) \right) \\
 &= \bigwedge_{z \in X} (\mathcal{U}^*(\top_{(x,z)}^*) \rightarrow f(z)) \\
 &= \mathcal{J}_{\mathcal{U}}(f)(x).
 \end{aligned}$$

(4) For each $x \in X, f \in L^X$,

$$\begin{aligned}
 &\mathcal{H}_{\mathcal{U}}(\mathcal{H}_{\mathcal{U}}(f))(x) \\
 &= \bigvee_{y \in X} (\mathcal{U}^*(\top_{(x,y)}^*) \odot \mathcal{H}_{\mathcal{U}}(f)(y)) \\
 &= \bigvee_{y \in X} (\mathcal{U}^*(\top_{(x,y)}^*) \odot \bigvee_{z \in X} (\mathcal{U}^*(\top_{(y,z)}^*) \odot f(z)))
 \end{aligned}$$

$$\begin{aligned}
 &= \bigvee_{z \in X} \left(\bigvee_{y \in X} \mathcal{U}^*(\top_{(x,y)}^*) \odot \mathcal{U}^*(\top_{(y,z)}^*) \odot f(z) \right) \\
 &= \bigvee_{z \in X} (\mathcal{U}^*(\top_{(x,z)}^*) \odot f(z)).
 \end{aligned}$$

Theorem 3.6. Let $(X, \mathcal{J})$ and $(X, \mathcal{H})$ be L -lower and L -upper approximation spaces. Define two maps $\mathcal{U}_{\mathcal{J}}, \mathcal{U}_{\mathcal{H}} : L^{X \times L} \longrightarrow L$ as

$$\begin{aligned}
 \mathcal{U}_{\mathcal{J}}(u) &= \bigwedge_{x,y \in X} (\mathcal{J}^*(\top_y^*)(x) \rightarrow u(x,y)), \\
 \mathcal{U}_{\mathcal{H}}(u) &= \bigwedge_{y \in X} (\mathcal{H}(\top_y)(x) \rightarrow u(x,y)).
 \end{aligned}$$

Then the following properties hold.

(1) $\mathcal{U}_{\mathcal{J}}$ is an Alexandrov L -fuzzy preuniformity such that $\mathcal{U}_{\mathcal{J}}(\top_{(x,y)}^*) = \mathcal{J}(\top_y^*)(x)$.

(2) $\mathcal{U}_{\mathcal{H}}$ is an Alexandrov L -fuzzy preuniformity such that $\mathcal{U}_{\mathcal{H}}(\top_{(x,y)}^*) = \mathcal{H}^*(\top_y)(x)$.

(3) If $\mathcal{J}$ is topological and there exists $x, y \in X$ with $\mathcal{J}(\top_y^*)(x) = \top$, then $\mathcal{U}_{\mathcal{J}}$ is an Alexandrov L -fuzzy quasi-uniformity.

(4) If $\mathcal{H}$ is topological and there exists $x, y \in X$ with $\mathcal{H}(\top_y)(x) = \perp$, then $\mathcal{U}_{\mathcal{H}}$ is an Alexandrov L -fuzzy quasi-uniformity.

(5) $\mathcal{U}_{\mathcal{J}_{\mathcal{U}}} = \mathcal{U}$ and $\mathcal{J}_{\mathcal{U}_{\mathcal{J}}} = \mathcal{J}$.

(6) $\mathcal{U}_{\mathcal{H}_{\mathcal{U}}} = \mathcal{U}$ and $\mathcal{H}_{\mathcal{U}_{\mathcal{H}}} = \mathcal{H}$.

Proof. (1) and (2). Since $\mathcal{J}^*(\top_x^*)(x) \geq \top_x(x) = \top$ and $\mathcal{H}(\top_x)(x) \geq \top_x(x) = \top$, it is similarly proved as Theorem 3.2.

(3) For each $x, z \in X$,

$$\begin{aligned}
 &\bigwedge_{y \in X} (\mathcal{J}^*(\top_z^*)(y) \rightarrow \mathcal{J}(\top_y^*)(x)) \\
 &= \mathcal{J}(\bigwedge_{y \in X} (\mathcal{J}^*(\top_z^*)(y) \rightarrow \top_y^*)(x)) \\
 &= \mathcal{J}(\mathcal{J}(\top_z^*)(x)) = \mathcal{J}(\top_z^*)(x).
 \end{aligned}$$

Then, $\bigvee_{y \in X} (\mathcal{J}^*(\top_z^*)(y) \odot \mathcal{J}^*(\top_y^*)(x)) = \mathcal{J}^*(\top_z^*)(x)$.

$$\begin{aligned}
 &\mathcal{U}_{\mathcal{J}}(v) \odot \mathcal{U}_{\mathcal{J}}(w) \\
 &= \bigwedge_{x,y \in X} (\mathcal{J}^*(\top_y^*)(x) \rightarrow v(x,y)) \\
 &\quad \odot \bigwedge_{z,w \in X} (\mathcal{J}^*(\top_w^*)(z) \rightarrow w(z,w)) \\
 &\leq \bigwedge_{x,y,z \in X} (\mathcal{J}^*(\top_y^*)(x) \rightarrow v(x,y))
 \end{aligned}$$

$$\begin{aligned}
 &\odot (\mathcal{J}^*(\top_w^*)(z) \rightarrow w(z,w)) \\
 &\leq \bigwedge_{x,y,z \in X} (\mathcal{J}^*(\top_y^*)(x) \odot \mathcal{J}^*(\top_z^*)(y)) \\
 &\quad \rightarrow v(x,y) \odot w(y,z)) \\
 &\leq \bigwedge_{x,z \in X} \left(\bigvee_{y \in X} (\mathcal{J}^*(\top_y^*)(x) \odot \mathcal{J}^*(\top_z^*)(y)) \right. \\
 &\quad \left. \rightarrow \bigvee_{y \in X} (v(x,y) \odot w(y,z)) \right) \\
 &\leq \bigwedge_{x,z \in X} (\mathcal{J}^*(\top_z^*)(x) \rightarrow u(x,z)) = \mathcal{U}(u).
 \end{aligned}$$

Hence $\mathcal{U}_{\mathcal{J}}(u) \geq \bigvee \{\mathcal{U}_{\mathcal{J}}(v) \odot \mathcal{U}_{\mathcal{J}}(w) \mid v \circ w \leq u\}$.

Since $\Delta_{X \times X} \circ u = u$ and

$$\begin{aligned}
 \mathcal{U}_{\mathcal{J}}(\Delta_{X \times X}) &= \bigwedge_{x,y \in X} (\mathcal{J}^*(\top_y^*)(x) \rightarrow \Delta_{X \times X}(x,y)) \\
 &= \bigvee_{x \neq y \in X} \mathcal{J}(\top_y^*)(x) = \top,
 \end{aligned}$$

we have $\bigvee \{\mathcal{U}_{\mathcal{J}}(v) \odot \mathcal{U}_{\mathcal{J}}(w) \mid v \circ w \leq u\} \geq \mathcal{U}_{\mathcal{J}}(\Delta_{X \times X}) \odot \mathcal{U}_{\mathcal{J}}(u) = \mathcal{U}_{\mathcal{J}}(u)$.

(4) It is similarly proved as (3).

(5) Since $\mathcal{J}_{\mathcal{U}}(\top_y^*)(x) = \bigwedge_{x,y \in X} (\mathcal{U}^*(\top_{(x,y)}^*) \rightarrow \top_y^*(y)) = \mathcal{U}(\top_{(x,y)}^*)$, we have

$$\begin{aligned}
 \mathcal{U}_{\mathcal{J}_{\mathcal{U}}}(u) &= \bigwedge_{x,y \in X} (\mathcal{J}_{\mathcal{U}}^*(\top_y^*)(x) \rightarrow u(x,y)) \\
 &= \bigwedge_{x,y \in X} (\mathcal{U}^*(\top_{(x,y)}^*) \rightarrow u(x,y)) \\
 &= \bigwedge_{x,y \in X} (u^*(x,y) \rightarrow \mathcal{U}(\top_{(x,y)}^*)) \\
 &= \mathcal{U}(\bigwedge_{x,y \in X} (u^*(x,y) \rightarrow \top_{(x,y)}^*)) \\
 &= \mathcal{U}(u).
 \end{aligned}$$

Since $\mathcal{U}_{\mathcal{J}}(\top_{(x,y)}^*) = \bigwedge_{x,y \in X} (\mathcal{J}^*(\top_y^*)(x) \rightarrow \top_{(x,y)}^*(x,y)) = \mathcal{J}(\top_y^*)(x)$, we have

$$\begin{aligned}
 \mathcal{J}_{\mathcal{U}_{\mathcal{J}}}(f)(x) &= \bigwedge_{y \in X} (\mathcal{U}_{\mathcal{J}}^*(\top_{(x,y)}^*) \rightarrow f(y)) \\
 &= \bigwedge_{y \in X} (\mathcal{J}^*(\top_y^*)(x) \rightarrow f(y)) \\
 &= \bigwedge_{y \in X} (f^*(y) \rightarrow \mathcal{J}(\top_y^*)(x)) \\
 &= \mathcal{J}(\bigwedge_{y \in X} (f^*(y) \rightarrow \top_y^*)(x)) \\
 &= \mathcal{J}(f)(x).
 \end{aligned}$$

(6) Since $\mathcal{H}_{\mathcal{U}}(\top_y)(x) = \bigvee_{y \in X} (\mathcal{U}^*(\top_{(x,y)}^*) \odot \top_y(y)) = \mathcal{U}(\top_{(x,z)}^*)$, we have $\mathcal{U}^*(\top_{(x,y)}^*)$, we have

$$\begin{aligned}\mathcal{H}_{\mathcal{U}}(u) &= \bigwedge_{x,y \in X} (\mathcal{H}_{\mathcal{U}}(\top_y)(x) \rightarrow u(x,y)) \\ &= \bigwedge_{x,y \in X} (\mathcal{U}^*(\top_{(x,y)}^*) \rightarrow u(x,y)) \\ &= \bigwedge_{x,y \in X} (u^*(x,y) \rightarrow \mathcal{U}(\top_{(x,y)}^*)) \\ &= \mathcal{U}(\bigwedge_{x,y \in X} (u^*(x,y) \rightarrow \top_{(x,y)}^*)) \\ &= \mathcal{U}(u).\end{aligned}$$

Since $\mathcal{U}_{\mathcal{H}}(\top_{(x,y)}^*) = \bigwedge_{y \in X} (\mathcal{H}(\top_y)(x) \rightarrow \top_{(x,y)}^*(x,y)) = \mathcal{H}^*(\top_y)(x)$, we have

$$\begin{aligned}\mathcal{H}_{\mathcal{H}}(f)(x) &= \bigvee_{y \in X} (\mathcal{U}_{\mathcal{H}}^*(\top_{(x,y)}^*) \odot f(y)) \\ &= \bigvee_{y \in X} (\mathcal{H}(\top_y)(x) \odot f(y)) \\ &= \mathcal{H}(\bigvee_{y \in X} (f(y) \odot \top_y))(x) \\ &= \mathcal{H}(f)(x).\end{aligned}$$

Theorem 3.7. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be Alexandrov L -fuzzy preuniform spaces. Let $\varphi : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ be L -fuzzy uniformly continuous. Then the following properties hold.

- (1) $\varphi : (X, R_{\mathcal{U}}) \rightarrow (Y, R_{\mathcal{V}})$ is an order preserving map.
- (2) $\varphi : (X, \mathcal{J}_{\mathcal{U}}) \rightarrow (Y, \mathcal{J}_{\mathcal{V}})$ is an L -lower approximation map.
- (3) $\varphi : (X, \mathcal{H}_{\mathcal{U}}) \rightarrow (Y, \mathcal{H}_{\mathcal{V}})$ is an L -upper approximation map.

Proof. (1) For any $x, z \in X$,

$$\begin{aligned}R_{\mathcal{U}}(x, z) &= \bigwedge_{u \in L^{X \times X}} (\mathcal{U}(u) \rightarrow u(x, y)) \\ &\leq \bigwedge_{(\varphi \times \varphi)^{\leftarrow}(v) \in L^{X \times X}} (\mathcal{U}((\varphi \times \varphi)^{\leftarrow}(v)) \\ &\quad \rightarrow (\varphi \times \varphi)^{\leftarrow}(v)(x, y)) \\ &\leq \bigwedge_{v \in L^{Y \times Y}} (\mathcal{V}(v) \rightarrow v(\varphi(x), \varphi(z))) \\ &= R_{\mathcal{V}}(\varphi(x), \varphi(z)).\end{aligned}$$

(2) Since $\mathcal{V}(\top_{(\varphi(x), \varphi(z))}^*) \leq \mathcal{U}((\varphi \times \varphi)^{\leftarrow}(\top_{(\varphi(x), \varphi(z))}^*)) \leq$

$$\begin{aligned}\mathcal{J}_{\mathcal{U}}(\varphi^{\leftarrow}(f))(x) &= \bigwedge_{z \in X} (\mathcal{U}^*(\top_{(x,z)}^*) \rightarrow \varphi^{\leftarrow}(f)(z)) \\ &\geq \bigwedge_{z \in X} (\mathcal{V}^*(\top_{(\varphi(x), \varphi(z))}^*) \rightarrow f(\varphi(z))) \\ &\geq \bigwedge_{y \in Y} (\mathcal{V}^*(\top_{(\varphi(x), y)}^*) \rightarrow f(y)) \\ &= \varphi^{\leftarrow}(\mathcal{J}_{\mathcal{V}}(f))(x).\end{aligned}$$

(3) Since $\mathcal{V}(\top_{(\varphi(x), \varphi(z))}^*) \leq \mathcal{U}((\varphi \times \varphi)^{\leftarrow}(\top_{(\varphi(x), \varphi(z))}^*)) \leq \mathcal{U}(\top_{(x,z)}^*)$, we have

$$\begin{aligned}\mathcal{H}_{\mathcal{U}}(\varphi^{\leftarrow}(f))(x) &= \bigvee_{z \in X} (\mathcal{U}^*(\top_{(x,z)}^*) \odot \varphi^{\leftarrow}(f)(z)) \\ &\leq \bigvee_{z \in X} (\mathcal{V}^*(\top_{(\varphi(x), \varphi(z))}^*) \odot f(\varphi(z))) \\ &\leq \bigvee_{y \in Y} (\mathcal{V}^*(\top_{(\varphi(x), y)}^*) \odot f(y)) \\ &= \varphi^{\leftarrow}(\mathcal{H}_{\mathcal{V}}(f))(x).\end{aligned}$$

Theorem 3.8. (1) Let $\varphi : (X, R_X) \rightarrow (Y, R_Y)$ be an order preserving map. Then $\varphi : (X, \mathcal{U}_{R_X}) \rightarrow (Y, \mathcal{U}_{R_Y})$ is L -fuzzy uniformly continuous.

(2) Let $\varphi : (X, \mathcal{J}_X) \rightarrow (Y, \mathcal{J}_Y)$ be an L -lower approximation map. Then $\varphi : (X, \mathcal{U}_{\mathcal{J}_X}) \rightarrow (Y, \mathcal{U}_{\mathcal{J}_Y})$ is L -fuzzy uniformly continuous.

(3) Let $\varphi : (X, \mathcal{H}_X) \rightarrow (Y, \mathcal{H}_Y)$ be an L -upper approximation map. Then $\varphi : (X, \mathcal{U}_{\mathcal{H}_X}) \rightarrow (Y, \mathcal{U}_{\mathcal{H}_Y})$ is L -fuzzy uniformly continuous.

Proof. (1) For any $x, z \in X$,

$$\begin{aligned}\mathcal{U}_{R_X}((\varphi \times \varphi)^{\leftarrow}(v)) &= \bigwedge_{x,y \in X} (R_X(x, y) \rightarrow (\varphi \times \varphi)^{\leftarrow}(v)(x, y)) \\ &\geq \bigwedge_{x,y \in X} (R_Y(\varphi(x), \varphi(y)) \rightarrow v(\varphi(x), \varphi(y))) \\ &\geq \bigwedge_{z,w \in Y} (R_Y(z, w) \rightarrow v(z, w)) = \mathcal{U}_{R_Y}(v).\end{aligned}$$

(2) Since $\varphi : (X, \mathcal{J}_X) \rightarrow (Y, \mathcal{J}_Y)$ is an L -lower approximation map, we have

$$\begin{aligned}\varphi^{\leftarrow}(\mathcal{J}_Y(\top_{\varphi(y)}^*)) &\leq \mathcal{J}_X(\varphi^{\leftarrow}(\top_{\varphi(y)}^*)) \leq \mathcal{J}_X(\top_y^*), \\ \mathcal{U}_{\mathcal{J}_X}((\varphi \times \varphi)^{\leftarrow}(v)) &= \bigwedge_{x,y \in X} (\mathcal{J}_X^*(\top_y^*)(x) \rightarrow (\varphi \times \varphi)^{\leftarrow}(v)(x, y))\end{aligned}$$

$$\begin{aligned} &\geq \bigwedge_{x,y \in X} (\mathcal{J}_Y^*(\top_{\varphi(y)}^*(\varphi(x)) \rightarrow v(\varphi(x), \varphi(y))) \\ &\geq \bigwedge_{z,w \in Y} (\mathcal{J}_Y^*(\top_z^*)(w) \rightarrow v(z, w)) = \mathcal{U}_{\mathcal{J}_Y}(v). \end{aligned}$$

(3) Since $\varphi : (X, \mathcal{H}_X) \rightarrow (Y, \mathcal{H}_Y)$ is an L -upper approximation map, we have $\mathcal{H}_Y(\top_{\varphi(y)})(\varphi(x)) = \varphi^{\leftarrow}(\mathcal{H}_Y(\top_{\varphi(y)})(x)) \geq \mathcal{H}_X(\varphi^{\leftarrow}(\top_{\varphi(y)})(x)) \geq \mathcal{H}_X(\top_y)(x)$.

$$\begin{aligned} &\mathcal{U}_{\mathcal{H}_X}((\varphi \times \varphi)^{\leftarrow}(v)) \\ &= \bigwedge_{x,y \in X} (\mathcal{H}_X(\top_y)(x) \rightarrow (\varphi \times \varphi)^{\leftarrow}(v)(x, y)) \\ &\geq \bigwedge_{x,y \in X} (\mathcal{H}_Y(\top_{\varphi(x)})(\varphi(y)) \rightarrow v(\varphi(x), \varphi(y))) \\ &\geq \bigwedge_{z,w \in Y} (\mathcal{H}_Y(\top_z)(w) \rightarrow v(z, w)) = \mathcal{U}_{\mathcal{H}_Y}(v). \end{aligned}$$

Definition 3.9 ([17, 20]). Suppose that $F : \mathcal{D} \rightarrow \mathcal{C}$, $G : \mathcal{C} \rightarrow \mathcal{D}$ are concrete functors.

(1) $\mathcal{C}$ and $\mathcal{D}$ are said to be *isomorphic* if $F \circ G = id_{\mathcal{C}}$ and $G \circ F = id_{\mathcal{D}}$.

(2) The pair (F, G) is called a *Galois correspondence* between $\mathcal{C}$ and $\mathcal{D}$ if for each $Y \in \mathcal{C}$, $id_Y : F \circ G(Y) \rightarrow Y$ is a $\mathcal{C}$ -morphism, and for each $X \in \mathcal{D}$, $id_X : X \rightarrow G \circ F(X)$ is a $\mathcal{D}$ -morphism.

If (F, G) is a Galois correspondence, then it is easy to check that F is a left adjoint of G , or equivalently that G is a right adjoint of F .

Let **AUS** be denote the category of Alexandrov L -fuzzy pre-uniform spaces and uniform continuous maps for morphisms. Let **FRS** be denote the category of L -fuzzy reflexive relations and order preserving maps for morphisms. Let **LAP** (resp. **UAP**) be denote the category of L -lower (resp. L -upper) approximation spaces and L -lower (resp. L -upper) approximation maps for morphisms.

Theorem 3.10. (1) $F : \mathbf{AUS} \rightarrow \mathbf{FRS}$ defined as $F(X, \mathcal{U}_X) = (X, R_{\mathcal{U}_X})$ is a functor.

(2) $G : \mathbf{FRS} \rightarrow \mathbf{AUS}$ defined as $G(X, R_X) = (X, \mathcal{U}_{R_X})$ is a functor.

(3) The pair (F, G) is a Galois correspondence between **AUS** and **FRS**.

(4) Two categories **AUS** and **LAP** are isomorphic.

(5) Two categories **AUS** and **UAP** are isomorphic.

Proof. (1) and (2) are follows from Theorems 3.2, 3.4, 3.7 and 3.8.

(3) By Theorem 3.4(4), if (X, R_X) is an L -fuzzy reflexive relation, then $F(G(X, R_X) = (X, R_{\mathcal{U}_{R_X}} = R_X)$. Hence, the identity map $id_X : (X, R_X) \rightarrow (X, R_{\mathcal{U}_{R_X}}) = F(G(X, R_X))$ is an order preserving map. Moreover, if $(X, \mathcal{U}_X)$ is an Alexandrov L -fuzzy pre-uniform space, then by Theorem 3.4(3), we have $G(F(X, \mathcal{U}_X)) = \mathcal{U}_{R_{\mathcal{U}_X}} \geq \mathcal{U}_X$. Hence the identity map $id_X : G(F(X, \mathcal{U}_X)) = \mathcal{U}_{R_{\mathcal{U}_X}} \rightarrow (X, \mathcal{U}_X)$ is uniform continuous. Therefore (F, G) is a Galois correspondence.

(4) By Theorems 3.5 and 3.7, $H : \mathbf{AUS} \rightarrow \mathbf{LAP}$ defined as $H(X, \mathcal{U}_X) = (X, \mathcal{J}_{\mathcal{U}_X})$ is a functor. By Theorems 3.6 and 3.8, $K : \mathbf{LAP} \rightarrow \mathbf{AUS}$ defined as $K(X, \mathcal{J}_X) = (X, \mathcal{U}_{\mathcal{J}_X})$ is a functor. By Theorem 3.5(6),

$$\begin{aligned} K(H(X, \mathcal{U}_X)) &= K(X, \mathcal{J}_{\mathcal{U}_X}) = (X, \mathcal{U}_{\mathcal{J}_{\mathcal{U}_X}} = \mathcal{U}_X), \\ H(K(X, \mathcal{J}_X)) &= H(X, \mathcal{U}_{\mathcal{J}_X}) = (X, \mathcal{J}_{\mathcal{U}_{\mathcal{J}_X}} = \mathcal{J}_X). \end{aligned}$$

Hence, two categories **AUS** and **LAP** are isomorphic.

(5) By Theorems 3.5 and 3.7, $P : \mathbf{AUS} \rightarrow \mathbf{UAP}$ defined as $P(X, \mathcal{U}_X) = (X, \mathcal{H}_{\mathcal{U}_X})$ is a functor. By Theorems 3.6 and 3.8, $Q : \mathbf{UAP} \rightarrow \mathbf{AUS}$ defined as $Q(X, \mathcal{H}_X) = (X, \mathcal{U}_{\mathcal{H}_X})$ is a functor. By Theorem 3.5(5),

$$\begin{aligned} Q(P(X, \mathcal{U}_X)) &= Q(X, \mathcal{H}_{\mathcal{U}_X}) = (X, \mathcal{U}_{\mathcal{H}_{\mathcal{U}_X}} = \mathcal{U}_X), \\ P(Q(X, \mathcal{H}_X)) &= P(X, \mathcal{U}_{\mathcal{H}_X}) = (X, \mathcal{H}_{\mathcal{U}_{\mathcal{H}_X}} = \mathcal{H}_X). \end{aligned}$$

Hence, two categories **AUS** and **UAP** are isomorphic. $\square$

Theorem 3.11. (1) Let $\mathcal{J} : L^X \rightarrow L^X$ be an L -lower approximation operator. Define $\mathcal{H}_{\mathcal{J}} : L^X \rightarrow L^X$ as $\mathcal{H}_{\mathcal{J}}(f) = \mathcal{J}^*(f^*)$. Then $\mathcal{H}_{\mathcal{J}}$ is an L -upper approximation operator.

(2) Let $\mathcal{H} : L^X \rightarrow L^X$ be an L -upper approximation operator. Define $\mathcal{J}_{\mathcal{H}} : L^X \rightarrow L^X$ as $\mathcal{J}_{\mathcal{H}}(f) = \mathcal{H}^*(f^*)$. Then $\mathcal{J}_{\mathcal{H}}$ is an L -lower approximation operator.

(3) Let $\mathcal{T} : L^X \rightarrow L$ be an Alexandrov L -fuzzy topology. Define $\mathcal{T}^s : L^X \rightarrow L$ as $\mathcal{T}^s(f) = \mathcal{T}(f^*)$. Then $\mathcal{T}^s$ is an Alexandrov L -fuzzy topology.

Proof. (1) For $f \in L^X$ and $\alpha \in L$, we have $\mathcal{H}_{\mathcal{J}}(f) = \mathcal{J}^*(f^*) \geq (f^*)^* = f$ and $\mathcal{H}_{\mathcal{J}}(\alpha \odot f) = \mathcal{J}^*((\alpha \odot f)^*) = \mathcal{J}^*(\alpha \rightarrow f^*) = (\alpha \rightarrow \mathcal{J}^*(f^*))^* = \alpha \odot \mathcal{J}^*(f^*) = \alpha \odot \mathcal{H}_{\mathcal{J}}(f)$. The remaining can be proved by similar way. $\square$

Lemma 3.12. For every $f \in L^X$, we define $u_f, u_f^{-1} : X \times X \rightarrow L$ by:

$$\begin{aligned} u_f(x, y) &= f(x) \rightarrow f(y), \\ u_f^{-1}(x, y) &= u_f(y, x). \end{aligned}$$

Then we have the following statements

- (1) $\top_{X \times X} = u_{\perp_X} = u_{\top_X}$.
- (2) For every $u_f \in L^{X \times X}$, we have $u_f \circ u_f \leq u_f$.
- (3) $u_f \odot u_g \leq u_{f \odot g}$, for each $f, g \in L^X$.
- (4) $u_f^{-1} = u_{f^*}$.
- (5) $u_{(\bigvee_{i \in \Gamma} f_i)} \geq \bigwedge_{i \in \Gamma} u_{f_i}$ and $u_{(\bigwedge_{i \in \Gamma} f_i)} \geq \bigwedge_{i \in \Gamma} u_{f_i}$.
- (6) $u_{\alpha \rightarrow f} \geq u_f$ and $u_{\alpha \odot f} \geq u_f$.
- (7) $u_f \geq \bigwedge_{z \in X} u_{\top_z}$ and $u_f \geq \bigwedge_{z \in X} u_{\top_z}^{-1}$.

Proof. (1) $\top_{X \times X}(x, y) = \top = u_{\perp_X}(x, y) = \perp_X(x) \rightarrow \perp_X(y) = \top_X(x) \rightarrow \top_X(y) = u_{\top_X}(x, y)$.

(2) For each $x, z \in X$,

$$\begin{aligned} (u_f \circ u_f)(x, z) &= \bigvee_{y \in X} u_f(x, y) \odot u_f(y, z) \\ &= \bigvee_{y \in X} ((f(x) \rightarrow f(y)) \odot (f(y) \rightarrow f(z))) \\ &\leq f(x) \rightarrow f(z) = u_f(x, z). \end{aligned}$$

(3) For each $x, y \in X$,

$$\begin{aligned} (u_f \odot u_g)(x, y) &= u_f(x, y) \odot u_g(x, y) \\ &= (f(x) \rightarrow f(y)) \odot (g(x) \rightarrow g(y)) \\ &\leq f(x) \odot g(x) \rightarrow f(y) \odot g(y) \\ &= u_{f \odot g}(x, y). \end{aligned}$$

(4)

$$\begin{aligned} u_f^{-1}(x, y) &= u_f(y, x) = f(y) \rightarrow f(x) = f^*(x) \rightarrow f^*(y) \\ &= u_{f^*}(x, y). \end{aligned}$$

(5) From Lemma 2.2(12), we have

$$\begin{aligned} u_{(\bigvee_{i \in \Gamma} f_i)}(x, y) &= \bigvee_{i \in \Gamma} f_i(x) \rightarrow \bigvee_{i \in \Gamma} f_i(y) \\ &\geq \bigwedge_{i \in \Gamma} (f_i(x) \rightarrow f_i(y)) = \bigwedge_{i \in \Gamma} u_{f_i}, \\ u_{(\bigwedge_{i \in \Gamma} f_i)} &= \bigwedge_{i \in \Gamma} f_i(x) \rightarrow \bigwedge_{i \in \Gamma} f_i(y) \\ &\geq \bigwedge_{i \in \Gamma} (f_i(x) \rightarrow f_i(y)) = \bigwedge_{i \in \Gamma} u_{f_i}. \end{aligned}$$

(6) From Lemma 2.2(10) and (11), we have

$$\begin{aligned} u_{\alpha \rightarrow f}(x, y) &= (\alpha \rightarrow f(x)) \rightarrow (\alpha \rightarrow f(y)) \\ &\geq f(x) \rightarrow f(y) = u_f(x, y), \\ u_{\alpha \odot f}(x, y) &= (\alpha \odot f(x)) \rightarrow (\alpha \odot f(y)) \\ &\geq f(x) \rightarrow f(y) = u_f(x, y). \end{aligned}$$

(7) Since $f = \bigvee_{z \in X} (f(z) \odot \top_z)$ and $f = \bigwedge_{z \in X} (f^*(z) \rightarrow \top_z^*)$, by (6) we have $u_f \geq \bigwedge_{z \in X} u_{\top_z}$ and $u_f \geq \bigwedge_{z \in X} u_{\top_z}^{-1}$.

Theorem 3.13. Let $\mathcal{U}$ be an Alexandrov L -fuzzy pre-uniformity on X . Define a map $\mathcal{T}_{\mathcal{U}} : L^X \rightarrow L$ as $\mathcal{T}_{\mathcal{U}}(f) = \mathcal{U}(u_f)$. Then the following properties hold.

- (1) $\mathcal{T}_{\mathcal{U}}$ is an Alexandrov L -fuzzy topology with $\mathcal{T}_{\mathcal{U}}^s = \mathcal{T}_{\mathcal{U}^s}$.
- (2) There exists $R_{\mathcal{U}} \in L^{X \times X}$ such that

$$\mathcal{T}_{\mathcal{U}}(f) = \bigwedge_{x, y \in X} (R_{\mathcal{U}}(x, y) \rightarrow (f(x) \rightarrow f(y))).$$

(3) $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_{\mathcal{J}_{\mathcal{U}}}$, where $\mathcal{T}_{\mathcal{J}_{\mathcal{U}}}(f) = S(f, \mathcal{J}_{\mathcal{U}}(f))$.

(4) $\mathcal{T}_{\mathcal{U}^s} = \mathcal{T}_{\mathcal{H}_{\mathcal{U}}}$, where $\mathcal{T}_{\mathcal{H}_{\mathcal{U}}}(f) = S(\mathcal{H}_{\mathcal{U}}(f), f)$.

Proof. (1) (AT1) $\mathcal{T}_{\mathcal{U}}(\top_X) = \mathcal{U}(u_{\top_X}) = \top$ and $\mathcal{T}_{\mathcal{U}}(\perp_X) = \mathcal{U}(u_{\perp_X}) = \top$.

(AT2) By Lemma 3.12(5), we have $\mathcal{T}_{\mathcal{U}}(\bigwedge_i f_i) = \mathcal{U}(u_{\bigwedge_i f_i}) \geq \bigwedge_i \mathcal{U}(u_{f_i}) = \bigwedge_i \mathcal{T}_{\mathcal{U}}(f_i)$, and $\mathcal{T}_{\mathcal{U}}(\bigvee_i f_i) = \mathcal{U}(u_{\bigvee_i f_i}) \geq \bigwedge_i \mathcal{U}(u_{f_i}) = \bigwedge_i \mathcal{T}_{\mathcal{U}}(f_i)$.

(AT3) By Lemma 3.12(6), we have $\mathcal{T}_{\mathcal{U}}(\alpha \odot f) = \mathcal{U}(u_{\alpha \odot f}) \geq \mathcal{U}(u_f) = \mathcal{T}_{\mathcal{U}}(f)$ and $\mathcal{T}_{\mathcal{U}}(\alpha \rightarrow f) = \mathcal{U}(u_{\alpha \rightarrow f}) \geq \mathcal{U}(u_f) = \mathcal{T}_{\mathcal{U}}(f)$. Hence, $\mathcal{T}_{\mathcal{U}}$ is an Alexandrov L -fuzzy topology on X . Moreover, $\mathcal{T}_{\mathcal{U}}^s(f) = \mathcal{T}_{\mathcal{U}}(f^*) = \mathcal{U}(u_{f^*}) = \mathcal{U}^s(u_f) = \mathcal{T}_{\mathcal{U}^s}(f)$.

(2) For $u_f = \bigwedge_{x, y \in X} (u_f^*(x, y) \rightarrow \top_{(x, y)}^*)$, there exists $R_{\mathcal{U}}(x, y) = \mathcal{U}^*(\top_{(x, y)}^*)$ such that

$$\begin{aligned} \mathcal{T}_{\mathcal{U}}(f) &= \mathcal{U}(u_f) \\ &= \mathcal{U}\left(\bigwedge_{x, y \in X} (u_f^*(x, y) \rightarrow \top_{(x, y)}^*)\right) \\ &= \bigwedge_{x, y \in X} (u_f^*(x, y) \rightarrow \mathcal{U}(\top_{(x, y)}^*)) \\ &= \bigwedge_{x, y \in X} (\mathcal{U}^*(\top_{(x, y)}^*) \rightarrow u_f(x, y)) \\ &= \bigwedge_{x, y \in X} (R_{\mathcal{U}}(x, y) \rightarrow (f(x) \rightarrow f(y))). \end{aligned}$$

(3)

$$\begin{aligned} \mathcal{T}_{\mathcal{J}_{\mathcal{U}}}(f) &= S(f, \mathcal{J}_{\mathcal{U}}(f)) \\ &= \bigwedge_{x \in X} \left(f(x) \rightarrow \mathcal{J}_{\mathcal{U}}(f)(x) \right) \\ &= \bigwedge_{x \in X} \left(f(x) \rightarrow \bigwedge_{z \in X} (R_{\mathcal{U}}(x, z) \rightarrow f(z)) \right) \\ &= \bigwedge_{x, z \in X} \left(f(x) \rightarrow (R_{\mathcal{U}}(x, z) \rightarrow f(z)) \right) \\ &= \bigwedge_{x, z \in X} \left(R_{\mathcal{U}}(x, z) \rightarrow (f(x) \rightarrow f(z)) \right) \end{aligned}$$

$$= \mathcal{T}_{\mathcal{U}}(f).$$

(4) From the fact $R_{\mathcal{U}^s}(y, x) = \mathcal{U}^{s*}(\top_{(y,x)}^*) = \mathcal{U}^*(\top_{(y,x)}^{-1*}) = \mathcal{U}^*(\top_{(x,y)}^*) = R(x, y)$,

$$\begin{aligned} \mathcal{T}_{\mathcal{H}_{\mathcal{U}}}(f) &= S(\mathcal{H}_{\mathcal{U}}(f), f) \\ &= \bigwedge_{x \in X} \left(\mathcal{H}_{\mathcal{U}}(f)(x) \rightarrow f(x) \right) \\ &= \bigwedge_{x \in X} \left(\left(\bigvee_{y \in X} (R_{\mathcal{U}}(x, y) \odot f(y)) \right) \rightarrow f(x) \right) \\ &= \bigwedge_{x, y \in X} \left(R_{\mathcal{U}}(x, y) \rightarrow (f(y) \rightarrow f(x)) \right) \\ &= \bigwedge_{x, z \in X} \left(R_{\mathcal{U}^s}(y, x) \rightarrow (f(y) \rightarrow f(x)) \right) \\ &= \mathcal{T}_{\mathcal{U}^s}(f). \end{aligned}$$

Example 3.14. Consider R defined in Example 3.3. By Theorem 3.13, we obtain two Alexandrov L -fuzzy topologies $\mathcal{T}_{\mathcal{U}_R}$, $\mathcal{T}_{\mathcal{U}_R^s} : L^X \rightarrow L$, where

$$\begin{aligned} \mathcal{T}_{\mathcal{U}_R}(f) &= \bigwedge_{x, y \in X} (R(x, y) \rightarrow (f(x) \rightarrow f(y))), \\ \mathcal{T}_{\mathcal{U}_R^s}(f) &= \mathcal{U}_R^s(u_f) = \mathcal{U}_R(u_{f*}) \\ &= \bigwedge_{x, y \in X} (R(x, y) \rightarrow (f^*(x) \rightarrow f^*(y))) \\ &= \bigwedge_{x, y \in X} (R(x, y) \rightarrow (f(y) \rightarrow f(x))). \end{aligned}$$

Theorem 3.15. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two L -fuzzy pre-uniform spaces. Let $\varphi : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ be L -fuzzy uniformly continuous. Then $\varphi : (X, \mathcal{T}_{\mathcal{U}}) \rightarrow (Y, \mathcal{T}_{\mathcal{V}})$ is L -fuzzy continuous.

Proof. For $x, y \in X$,

$$\begin{aligned} (\varphi \times \varphi)^{\leftarrow}(u_f)(x, y) &= u_f(\varphi(x), \varphi(y)) \\ &= f(\varphi(x)) \rightarrow f(\varphi(y)) \\ &= \varphi^{\leftarrow}(f)(x) \rightarrow \varphi^{\leftarrow}(f)(y) \\ &= u_{\varphi^{\leftarrow}(f)}(x, y). \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{T}_{\mathcal{U}}(\varphi^{\leftarrow}(f)) &= \mathcal{U}(u_{\varphi^{\leftarrow}(f)}) \\ &= \mathcal{U}((\varphi \times \varphi)^{\leftarrow}(u_f)) \geq \mathcal{V}(u_f) = \mathcal{T}_{\mathcal{V}}(f). \end{aligned}$$

4. Conclusion

We obtained an adjointness between the category of Alexandrov L -fuzzy pre-uniform spaces and reflexive L -fuzzy relations. In addition, we showed that the category of Alexandrov L -fuzzy pre-uniform spaces and the category of L -lower (resp. upper) approximation spaces are isomorphic.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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