On Jensen-Type and Hölder-Type Inequality for Interval-Valued Choquet Integrals

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Abstract

Wang (Journal of Applied Mathematics and Computing, vol. 35, no. 1-2, pp. 305-321, 2011) studied Jensen-type and Hölder-type inequality for Choquet integral. In this paper, we consider the interval-valued Choquet integral with respect to a fuzzy measure and investigate Jensen-type and Hölder-type inequality for interval-valued Choquet integrals.

Keywords: Choquet integral, Jensen-type and Hölder-type inequality, Interval-valued function

1. Introduction

The Choquet integrals have been studied by many researchers (see [1–6]). Aumann [7], Jang and his colleagues [8–13], and Zhang et al. [14] also have been studying the interval-valued Choquet integrals which are related with some properties and applications of them. Various integral inequalities, such as Jensen's inequality, Hölder's inequality, Minkowski's inequality, and Chebyshev's inequality for some integrals were developed by the authors in [3, 5, 15, 16].

The main idea of this paper is to prove two-types integral inequalities of the interval-valued Choquet integral which was defined by Jang and his colleagues [8, 11] and Zhang et al. [14].

In Section 2, we list definitions and basic properties of the Choquet integral with respect to a fuzzy measure, and introduce Jensen's and Hölder's inequalities of them. In Section 3, we list some definitions of intervals on $[0, 1]$ and the interval-valued Choquet integral, and introduce some properties of the interval-valued Choquet integral.

In Section 4, we investigate Jensen's and Hölder's inequality for interval-valued Choquet integrals.

2. Preliminaries and Definitions

In this section, we give some definitions and basic properties of the Choquet integral as follows:

Definition 2.1 ([2, 4, 5]). Let X be a nonempty set, Ω be a σ -algebra of subsets of X , and $\mu : \Omega \rightarrow [0, \infty)$ be a nonnegative real-valued set function. μ is said to be a fuzzy measure if

- (1) $\mu(\emptyset) = 0$;
- (2) for any $A, B \in \Omega$, $A \subseteq B$ implies $\mu(A) \leq \mu(B)$;

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(3) for $\{A_n\} \subset \Omega$, $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq \dots$ implies $\lim_{n \rightarrow \infty} \mu(A) = \mu(\bigcup_{n=1}^{\infty} A_n)$ (continuity from below);
 (4) for $\{A_n\} \subset \Omega$, $A_1 \supseteq A_2 \supseteq \dots \supseteq A_n \supseteq \dots$ and $\mu(A_1) < \infty$ implies $\lim_{n \rightarrow \infty} \mu(A_n) = \mu(\bigcap_{n=1}^{\infty} A_n)$ (continuity from above).

Note that (X, Ω, μ) is called a fuzzy measure space. Let $F(X)$ be the set of all real-valued nonnegative measurable functions defined on X .

Definition 2.2. Let (X, Ω, μ) be a fuzzy measure. μ is said to be submodular if

$$\mu(A \cap B) + \mu(A \cup B) \leq \mu(A) + \mu(B)$$

for any $A, B \in \Omega$.

Definition 2.3. Let $f, g \in F(X)$. f and g are said to be comonotonic if and only if

$$f(x) < f(x') \text{ implies } g(x) \leq g(x')$$

for any $x, x' \in X$.

Definition 2.4. Let (X, Ω, μ) be a fuzzy measure space, $f \in F(X)$ and $A \in \Omega$.

(1) The Choquet integral of f with respect to μ on A is defined by

$$(C) \int_A f d\mu = \int_0^\infty \mu(A \cap \{x \mid f(x) \geq \alpha\}) d\alpha.$$

- (2) If $(C) \int_X f d\mu < \infty$, then f is called (C) -integrable.
 (3) $L_1(\mu)$ is the set of all (C) -integrable functions.

Theorem 2.1 ([2–6, 8–14, 16]). Let (X, Ω, μ) be a fuzzy measure space, $\{f_1, f_2, f\} \subset F(X)$, $A, B \in \Omega$ and c be a nonnegative real number. Then, the Choquet integral has the following properties:

- (1) If $\mu(A) = 0$, then $(C) \int_A f d\mu = 0$;
 (2) $(C) \int_A c d\mu = c\mu(A)$;
 (3) If $f_1 \leq f_2$, then $(C) \int_A f_1 d\mu \leq (C) \int_A f_2 d\mu$;
 (4) If $A \subset B$, then $(C) \int_A f d\mu \leq (C) \int_B f d\mu$;
 (5) $(C) \int_A (f + c) d\mu = (C) \int_A f d\mu + c \cdot \mu(A)$;
 (6) $(C) \int_A c \cdot f d\mu = c \cdot (C) \int_A f d\mu$.

Theorem 2.2 ([5]). Let (X, Ω, μ) be a fuzzy measure space and $f, g \in F(X)$.

- (1) If f and g are comonotonic, then for $A \in \Omega$,

$$(C) \int_A (f + g) d\mu = (C) \int_A f d\mu + (C) \int_A g d\mu.$$

- (2) If μ is submodular, then for any $A \in \Omega$,

$$(C) \int_A (f + g) d\mu \leq (C) \int_A f d\mu + (C) \int_A g d\mu.$$

Theorem 2.3 (Markov-type inequality [5]). Let (X, Ω, μ) be a fuzzy measure space and $f \in F(X)$. If c is a positive real number, then

$$\mu(A \cap \{x : f(x) \geq c\}) \leq \frac{1}{c} (C) \int_A f d\mu$$

for any $A \in \Omega$.

Theorem 2.4 (Jensen-type inequality [5]). Let (X, Ω, μ) be a fuzzy measure space and $f \in L_1(\mu)$. If μ is a regular fuzzy measure and $\phi : [0, \infty) \rightarrow [0, \infty)$ is a convex function, then

$$\phi\left((C) \int_X f d\mu\right) \leq (C) \int_X \phi(f) d\mu.$$

Theorem 2.5 (Hölder-type inequality [5]). Let (X, Ω, μ) be a fuzzy measure and $f, g \in F(X)$. If μ is a submodular fuzzy measure and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$(C) \int_X f g d\mu \leq \left((C) \int_X f^p d\mu\right)^{\frac{1}{p}} \left((C) \int_X g^q d\mu\right)^{\frac{1}{q}}.$$

3. Interval-Valued Choquet Integrals

In this section, we introduce the interval-valued Choquet integral with respect to a fuzzy measure on the space $IF(X)$ of all interval-valued measurable functions as follows [2, 8–14].

A set-valued function is a mapping G from X to $\wp(\mathbb{R}^+)$. Let $\mathbb{R}^+ = [0, \infty)$ and

$$I(\mathbb{R}^+) = \{\bar{a} = [a^-, a^+] \mid a^-, a^+ \in \mathbb{R}^+ \text{ and } a^- \leq a^+\}.$$

$IF(X)$ be the space of all interval-valued functions from X to $I(\mathbb{R}^+)$.

Definition 3.1. If $\bar{a}, \bar{b} \in I(\mathbb{R}^+)$, $k \in \mathbb{R}^+$, then we define the following operations:

- (1) $\bar{a} + \bar{b} = [a^- + b^-, a^+ + b^+]$,
 (2) $k\bar{a} = [ka^-, ka^+]$,
 (3) $\bar{a} \wedge \bar{b} = [a^- \wedge b^-, a^+ \wedge b^+]$,
 (4) $\bar{a} \vee \bar{b} = [a^- \vee b^-, a^+ \vee b^+]$,
 (5) $\bar{a} \leq \bar{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$,
 (6) $\bar{a} < \bar{b}$ if and only if $\bar{a} \leq \bar{b}$ and $\bar{a} \neq \bar{b}$,
 (7) $\bar{a} \subset \bar{b}$ if and only if $b^- \leq a^-$ and $a^+ \leq b^+$.

Definition 3.2. Let G be a set-valued function and $A \in \Omega$. Then the Choquet integral of G on A is defined by

$$(C) \int_A G d\mu = \left\{ (C) \int_A g d\mu \mid g \in S(G) \right\},$$

where $S(G)$ is the family of μ -a.e. measurable selections of G .

Recall that instead of $(C) \int_X G d\mu$, we write $(C) \int_X G d\mu$ and a set-valued function G is said to be integral existing if $(C) \int_X G d\mu \neq \emptyset$, and G is said to be Choquet integrable if $(C) \int F d\mu$ exists and does not include ∞ .

Definition 3.3 ([9]). A set-valued function G is said to be Choquet integrable bounded if there is a Choquet integrable function h such that

$$\|G(x)\| \sup_{r \in G(x)} |r| \leq h(x).$$

Theorem 3.1 ([9]). If a set-valued function G is Choquet integrably bounded, then G is Choquet integrable.

Theorem 3.2 ([9]). (1) If G is a Choquet integrable set-valued function and $A, B \in \Omega$ with $A \subset B$, then

$$(C) \int_A G d\mu \leq (C) \int_B G d\mu.$$

(2) If G_1 and G_2 are Choquet integrably bounded set-valued functions and $G_1 \leq G_2$, then

$$(C) \int G_1 d\mu \leq (C) \int G_2 d\mu.$$

(3) If G is a Choquet integrable set-valued function, then

$$(C) \int aG d\mu = a(C) \int G d\mu$$

for all $a \geq 0$.

Theorem 3.3 ([9, 14]). If $G = [g^-, g^+] \in IF(X)$ is a measurable and Choquet integrably bounded, then

$$(C) \int G d\mu = \left[(C) \int g^- d\mu, (C) \int g^+ d\mu \right].$$

4. Three-Types Inequalities for the Interval-Valued Choquet Integrals

In this section, we investigate Markov-type inequality, Jensen-type inequality and Hölder-type inequality for the interval-valued Choquet integrals. By Definition 3.3, we have that if

$G = [g^-, g^+] \in IF(X)$ and G is Choquet integrably bounded, then there exists Choquet integrable function h such that $g^- \leq g^+ \leq h$ and

$$(C) \int G d\mu = \left[(C) \int g^- d\mu, (C) \int g^+ d\mu \right] \in I(\mathbb{R}^+).$$

Theorem 4.1 (Markov-type inequality for the interval-valued Choquet integrals). Let (X, Ω, μ) be a fuzzy measure space. If $G \in IF(X)$ is integrably bounded and c is a positive real number, then

$$\mu(A \cap \{x \mid G(x) \geq c\}) \leq \frac{1}{c} (C) \int_A G d\mu.$$

Proof. Note that if $G(x) = [g^-(x), g^+(x)] \geq c$, then $g^-(x) \geq c$ and $g^+(x) \geq c$. By Theorem 2.3, we get

$$\mu(A \cap \{x \mid g^-(x) \geq c\}) \leq \frac{1}{c} (C) \int_A g^- d\mu, \quad (1)$$

and

$$\mu(A \cap \{x \mid g^+(x) \geq c\}) \leq \frac{1}{c} (C) \int_A g^+ d\mu, \quad (2)$$

for any $A \in \Omega$. Since $g^-(x) \leq g^+(x)$ and μ is monotonic, we get

$$\mu(A \cap \{x \mid g^-(x) \geq c\}) \leq \mu(A \cap \{x \mid g^+(x) \geq c\}). \quad (3)$$

By Eqs. (1), (2), (3), and Theorem 2.3, we get

$$\begin{aligned} & \mu(A \cap \{x \mid G(x) = [g^-(x), g^+(x)] \geq c\}) \\ & \leq \mu(A \cap \{x \mid g^-(x) \geq c\}) \\ & \leq [\mu(A \cap \{x \mid g^-(x) \geq c\}), \mu(A \cap \{x \mid g^+(x) \geq c\})] \\ & \leq \left[\frac{1}{c} (C) \int_A g^- d\mu, \frac{1}{c} (C) \int_A g^+ d\mu \right] \\ & = \frac{1}{c} \left[(C) \int_A g^- d\mu, (C) \int_A g^+ d\mu \right] \\ & = \frac{1}{c} (C) \int_A G d\mu. \end{aligned} \quad \square$$

We denote that

$$\begin{aligned} IL_1(\mu) \\ = \{G \in IF(X) \mid G = [g^-, g^+] \text{ and } g^-, g^+ \in L_1(\mu)\}. \end{aligned}$$

Recall that $\phi : [0, \infty) \rightarrow [0, \infty)$ is a convex function, that is, $\phi(ax + by) \leq a\phi(x) + b\phi(y)$ for any $x, y \in [0, \infty)$ any positive constant a, b . We note that $\Phi(u, v) = [\phi^-(u), \phi^+(v)]$

is called a convex function if and only if ϕ^- and ϕ^+ are convex functions.

Theorem 4.2 (Jensen-type inequality for the interval-valued Choquet integrals). Let (X, Ω, μ) be a fuzzy measure space. If $G \in IL_1(\mu)$ is Choquet integrably bounded, μ is a regular fuzzy measure and

$$\Phi = [\phi^-, \phi^+] : [0, \infty) \times [0, \infty) \rightarrow [0, \infty) \times [0, \infty)$$

by $\Phi(u, v) = [\phi^-(u), \phi^+(v)]$ is a convex function, then

$$\Phi \circ \left((C) \int_X G d\mu \right) \leq (C) \int_X \Phi \circ G d\mu,$$

where $\Phi \circ [a, b] = [\phi^-(a), \phi^+(b)]$ and

$$\Phi \circ G = [\phi^-(g^-), \phi^+(g^+)].$$

Proof. Since $G \in IL_1(\mu)$, $g^-, g^+ \in L_1(\mu)$ and ϕ^-, ϕ^+ are convex functions. By Theorem 2.4, we get

$$\phi^- \left((C) \int_X g^- d\mu \right) \leq (C) \int_X \phi^-(g^-) d\mu, \quad (4)$$

and

$$\phi^+ \left((C) \int_X g^+ d\mu \right) \leq (C) \int_X \phi^+(g^+) d\mu. \quad (5)$$

By Eqs. (4), (5), and Theorem 2.4, we get

$$\begin{aligned} & \Phi \circ \left((C) \int_X G d\mu \right) \\ &= \Phi \circ \left(\left[(C) \int_X g^- d\mu, (C) \int_X g^+ d\mu \right] \right) \\ &= \left[\phi^- \left((C) \int_X g^- d\mu \right), \phi^+ \left((C) \int_X g^+ d\mu \right) \right] \\ &\leq \left[(C) \int_X \phi^-(g^-) d\mu, (C) \int_X \phi^+(g^+) d\mu \right] \\ &= (C) \int_X [\phi^-(g^-), \phi^+(g^+)] d\mu \\ &= (C) \int_X \Phi \circ G d\mu. \end{aligned} \quad \square$$

Theorem 4.3 (Hölder-type inequality for the interval-valued Choquet integrals). Let (X, Ω, μ) be a fuzzy measure and $G = [g^-, g^+]$, $H = [h^-, h^+] \in IF(X)$ be Choquet integrably bounded. If μ is a submodular fuzzy measure and $1 < p$,

$q < \infty$, $p, q \in \mathbb{N}$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$(C) \int_X GH d\mu \leq \left((C) \int_X G^p d\mu \right)^{\frac{1}{p}} \left((C) \int_X H^q d\mu \right)^{\frac{1}{q}}.$$

where $GH = [g^-h^-, g^+h^+]$, $G^p = [g^{-p}, g^{+p}]$, and $H^p = [h^{-p}, h^{+p}]$.

Proof. Note that if G and H are Choquet integrably bounded then GH is Choquet integrably bounded. Since $GH = [g^-h^-, g^+h^+]$, by Theorem 3.3, we have

$$(C) \int_X GH d\mu = \left[(C) \int_X g^-h^- d\mu, (C) \int_X g^+h^+ d\mu \right]. \quad (6)$$

By Theorem 2.5, we have

$$(C) \int_X g^-h^- d\mu \leq \left((C) \int_X (g^-)^p d\mu \right)^{\frac{1}{p}} \left((C) \int_X (h^-)^q d\mu \right)^{\frac{1}{q}}, \quad (7)$$

and

$$(C) \int_X g^+h^+ d\mu \leq \left((C) \int_X (g^+)^p d\mu \right)^{\frac{1}{p}} \left((C) \int_X (h^+)^q d\mu \right)^{\frac{1}{q}}. \quad (8)$$

Note that $[a^{\frac{1}{p}}, b^{\frac{1}{p}}]^p = [a, b]$ and hence $[a^{\frac{1}{p}}, b^{\frac{1}{p}}] = [a, b]^{\frac{1}{p}}$. By Eqs. (6), (7), and (8), we get

$$\begin{aligned} & (C) \int_X GH d\mu \\ &\leq \left[\left((C) \int_X (g^-)^p d\mu \right)^{\frac{1}{p}} \left((C) \int_X (h^-)^q d\mu \right)^{\frac{1}{q}}, \right. \\ &\quad \left. \left((C) \int_X (g^+)^p d\mu \right)^{\frac{1}{p}} \left((C) \int_X (h^+)^q d\mu \right)^{\frac{1}{q}} \right] \\ &= \left[\left((C) \int_X (g^-)^p d\mu \right)^{\frac{1}{p}} \left((C) \int_X (g^+)^p d\mu \right)^{\frac{1}{p}} \right. \\ &\quad \times \left. \left[\left((C) \int_X (h^-)^q d\mu \right)^{\frac{1}{q}} \left((C) \int_X (h^+)^q d\mu \right)^{\frac{1}{q}} \right] \right] \\ &= \left[(C) \int_X (g^-)^p d\mu, (C) \int_X (g^+)^p d\mu \right]^{\frac{1}{p}} \\ &\quad \times \left[(C) \int_X (h^-)^q d\mu, (C) \int_X (h^+)^q d\mu \right]^{\frac{1}{q}} \\ &= \left((C) \int_X [(g^-)^p, (g^+)^p] d\mu \right)^{\frac{1}{p}} \left((C) \int_X [(h^-)^q, (h^+)^q] d\mu \right)^{\frac{1}{q}} \end{aligned}$$

$$= \left((C) \int G^p d\mu \right)^{\frac{1}{p}} \left((C) \int H^q d\mu \right)^{\frac{1}{q}}. \quad \square$$

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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