Non-parametric Statistical Tests for Fuzzy Observations: Fuzzy Test Statistic Approach

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Abstract

A general approach to the problem of testing statistical non-parametric tests is proposed, for the case when the available data are fuzzy and the level of significance is given as a fuzzy number. To do this, the usual concepts of test statistic and critical value are extended to the fuzzy test statistic and fuzzy critical value, by using the α -cuts approach. The method of decision making (to accept or reject the hypothesis of interest) is based on a suitable ranking method. A numerical example is prepared to clarify the proposed approach.

Keywords: Location problem, scale problem, Fuzzy critical value, Fuzzy significance level, Preference degree

1. Introduction

Non-parametric approaches, including non-parametric tests, provide inferential procedures to statistics based on some weak assumptions regarding the nature of the underlying population distributions. A particular class of non-parametric tests is composed of two-sample tests. Such tests are commonly based on crisp (exact/nonfuzzy) observations. But, in real world, there are many situations in which the available data are imprecise (vague/fuzzy) rather than precise (crisp). For instance, in the source water studies, the water level of a river cannot be measured in an exact way because of the fluctuation. In this case, the level of water may be reported as imprecise quantities such as: “about 180 (cm)”, “approximately 145 (cm)”, etc. As another example, in survival analysis, we may not determine an exact value for the lifetime of a certain virus. A virus may active completely over a certain period but losing in effect for some time, and finally go dead completely at a certain time. In such case, we may report the lifetimes as imprecise quantities such as: “approximately 40 (h)”, “approximately 55 (h)”, and the like. To perform suitable statistical methods for dealing with imprecise observations, first we need to model such data, and then, extend the usual approach to imprecise environment. Fuzzy set theory seems to have suitable tools for modeling these data and providing appropriate statistical methods based on such data.

After introducing fuzzy set theory, there have been a lot of attempts for developing fuzzy statistical methods. But, as the authors know, there have been a few works on non-parametric approach in fuzzy environment. Concerning the purposes of this article, let us briefly review some of the literature on this topic. Kahranam et al. [1] proposed some algorithms for fuzzy non-parametric rank-sum tests based on fuzzy random variables.

Grzegorzewski [2] introduced a method for inference about the median of a population

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using fuzzy random variables.

Also, the author demonstrated a straightforward generalization of some classical non-parametric tests for fuzzy random variables [3]. The last work relies on the quasi-ordering based on a metric in the space of fuzzy numbers. He studied some non-parametric median tests based on the necessity index of strict dominance suggested by Dubios and Prade [4], for fuzzy observations [5, 6]. In this manner, he obtained a fuzzy test showing a degree of possibility and a degree of necessity for evaluating the underlying hypotheses. Denoeux et al. [7], using a fuzzy partial ordering on closed intervals, extended the non-parametric rank-sum tests based on fuzzy data. For evaluating the hypotheses of interest, they employed the concepts of the fuzzy p -value and degree of rejection of the null hypothesis quantified by a degree of possibility and a degree of necessity, when a given significance level is a crisp number or a fuzzy set. Hryniewicz investigated the fuzzy version of the Goodman-Kruskal γ statistic described by ordered categorical data [8], see also [9]. Grzegorzewski and Szymanowski [10] studied the problem of Goodness-of-fit tests for fuzzy data. Taheri and Hesamian [11] developed the Wilcoxon-rank test for fuzzy data. They also studied some linear rank tests for fuzzy data, by using a p -value-based method [12]. Recently, Taheri et al. [13] studied the statistical inference for contingency tables when the available data are fuzzy rather than crisp.

The present paper aims to develop some non-parametric statistical two-sample tests for fuzzy data, based on extending the concept of classical critical value and comparing it with corresponding test statistic. This paper is organized as follows: In Section 2, we recall some concepts of fuzzy numbers and fuzzy random variables. In Section 3, based on an index for ranking fuzzy numbers, we introduce a method to construct a version of linear rank tests for fuzzy data. Also, we extend the concept of classical critical value when the given significance level is a fuzzy number, too. In Section 4, we provide the method of decision making to accept or reject the hypothesis of interest. To do this, we use a preference degree to compare the observed fuzzy test statistic and the fuzzy critical value. A numerical example is given in Section 5 to clarify the proposed approach. Finally, a brief conclusion is provided in Section 6. A brief review on some linear rank tests is given in Appendix.

2. Fuzzy Numbers and Fuzzy Random Variables

A fuzzy set $\tilde{A}$ of the universal set $\mathbf{X}$ is defined by its membership function $\mu_{\tilde{A}} : \mathbf{X} \rightarrow [0, 1]$, with the set $\text{supp}(\tilde{A}) =$

$\{x \in \mathbf{X} : \mu_{\tilde{A}}(x) > 0\}$, the support of $\tilde{A}$. We say $\tilde{A}$ is a normal fuzzy set if there exists at least one element $x \in \mathbf{X}$, such that $\mu_{\tilde{A}}(x) = 1$. In this work, we consider $\mathbb{R}$ (the real line) as the universal set. We denote by $\tilde{A}[\alpha]$ the α -cut of the fuzzy set $\tilde{A}$, defined for every $\alpha \in (0, 1]$, by $\tilde{A}[\alpha] = \{x \in \mathbb{R} : \mu_{\tilde{A}}(x) \geq \alpha\}$, and $\tilde{A}[0]$ is the closure of $\text{supp}(\tilde{A})$. The sequence $\{\tilde{A}[\alpha] : \alpha \in [0, 1]\}$ is a set representation of $\tilde{A}$. $\tilde{A}$ is called convex if $\mu_{\tilde{A}}(\lambda x + (1 - \lambda)y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$, $\forall \lambda \in [0, 1]$. The fuzzy set $\tilde{A}$ of $\mathbb{R}$ is called a fuzzy number if it is normal and convex, and for every $\alpha \in (0, 1]$, the set $\tilde{A}[\alpha]$ is a closed interval. Such an interval is denoted by $\tilde{A}[\alpha] = [\tilde{A}_{\alpha}^L, \tilde{A}_{\alpha}^U]$, where $\tilde{A}_{\alpha}^L = \inf\{x : x \in \tilde{A}[\alpha]\}$ and $\tilde{A}_{\alpha}^U = \sup\{x : x \in \tilde{A}[\alpha]\}$.

One of the very realistic kind of fuzzy numbers is the triangular fuzzy number denoted by $\tilde{A} = (\epsilon_1, a, \epsilon_2)_T$, with the following membership function

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - \epsilon_1}{a - \epsilon_1} & \text{if } \epsilon_1 \leq x < a, \\ \frac{\epsilon_2 - x}{\epsilon_2 - a} & \text{if } a \leq x \leq \epsilon_2, \\ 0 & \text{if } x < \epsilon_1 \text{ or } x > \epsilon_2. \end{cases}$$

We denote by $\mathbb{F}(\mathbb{R})$ the set of all fuzzy real numbers (for more on fuzzy numbers see [14]).

Note that, given a real number z , we can induce a fuzzy number $\tilde{z}$ with membership function $\mu_{\tilde{z}}(r)$ such that $\mu_{\tilde{z}}(x) = 1$ and $\mu_{\tilde{z}}(r) < 1$ for $r \neq x$ [15].

Let $\mathcal{F}(\mathbb{R}) \subseteq \mathbb{F}(\mathbb{R})$ be the set of all real fuzzy numbers induced by the real set $\mathbb{R}$. We define the relation $\sim$ on $\mathcal{F}(\mathbb{R})$ as $\tilde{z}_1 \sim \tilde{z}_2$ iff $\tilde{z}_1$ and $\tilde{z}_2$ are induced by the same real number z . Then $\sim$ is an equivalence classes $[\tilde{z}] = \{\tilde{a} : \tilde{a} \sim \tilde{z}\}$. The quotient set $\mathcal{F}(\mathbb{R}) / \sim$ is the set of all equivalence classes. We call $\mathcal{F}(\mathbb{R}) / \sim$ as the fuzzy real number system. In practice, we take only one element $\tilde{z}$ from each equivalence class $[\tilde{z}]$ to form the fuzzy real number system $(\mathcal{F}(\mathbb{R}) / \sim)_{\mathbb{R}}$ that is

$$(\mathcal{F}(\mathbb{R}) / \sim)_{\mathbb{R}} = \{\tilde{z} : \tilde{z} \text{ is the only element from } [\tilde{z}]\}.$$

To treat imprecise observations, we use the concept of fuzzy random variable similar to those of Grzegorzewski [6] and Wu [15].

Definition 2.1 Let $\mathcal{S}_X$ be the sample space of a random variable X defined on the probability space $(\Omega, \mathbb{A}, \mathbf{P})$. A fuzzy random variable is a mapping $\tilde{X} : \Omega \rightarrow (\mathcal{F}(\mathbb{S}_X) / \sim)_{\mathcal{S}_X}$ if it satisfies the following conditions

- (a) For any $\alpha \in [0, 1]$ and all $\omega \in \Omega$, the real valued mapping $\inf \tilde{X}_{\alpha} : \Omega \rightarrow \mathbb{R}$, satisfying $\inf \tilde{X}_{\alpha}(\omega) = \inf(\tilde{X}(\omega))_{\alpha}$ and $\sup \tilde{X}_{\alpha} : \Omega \rightarrow \mathbb{R}$ satisfying $\sup \tilde{X}_{\alpha}(\omega) = \sup(\tilde{X}(\omega))_{\alpha}$

real-valued random variables.

(b) $\{\tilde{X}_\alpha(\omega) : \alpha \in [0, 1]\}$ is a set representation of $\tilde{X}(\omega)$ for all $\omega \in \Omega$.

Remark 2.1 Note that, in the above definition, $(\mathcal{F}(\mathcal{S}_X)/\sim)_{\mathcal{S}_X}$ is the support of the fuzzy random variable $\tilde{X}$. Therefore, each α -cut of $\tilde{X}$ depends on the random variable X . Thus, the crisp random variable X may interpret as the origin of the fuzzy random variable $\tilde{X}$.

Definition 2.2 (see also [2]) We say that $\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_m$ (with observed $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_m$) are independent and identically distributed (briefly, say a fuzzy random sample) if their related origins $X_1, X_2, \dots, X_n$ are independent and identically distributed crisp random variables.

3. Linear Rrank Tests for Fuzzy Observations

In this section, we are going to extend the statistical linear tests to examine the hypothesis test about the differences in location or variability between two populations based on a set of imprecise (fuzzy) observations. The proposed approach is based on two key concepts of fuzzy test statistic and fuzzy critical value.

3.1 Fuzzy Test Statistic

First, note that the classical linear rank statistic can be rewritten as follows (see Appendix)

$$\begin{aligned} T_N &= \sum_{i=1}^n a \left[\sum_{j=1}^N I(x_i \geq z_j) \right] \\ &= \sum_{i=1}^n a \left[N - \sum_{j=1}^N I(z_j > x_i) \right], \end{aligned}$$

where, I denotes the indicator functions and z_j denotes the j th observation in combined observations $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_m$.

Now, to perform the non-parametric two-sample tests for location or scale problem based on fuzzy observations $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ and $\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_m$, we need a suitable method of ranking fuzzy numbers. Here, we recall a definition of a common method of ranking fuzzy numbers called the necessity index of strict dominance (NSD index), suggested by Dubois and Prade [4].

Definition 3.1 For two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, we can evaluate the degree of necessity to which the relation $\tilde{A} \succ \tilde{B}$ is fulfilled

Figure 1. Classical critical value.

by

$$Nec(\tilde{A} \succ \tilde{B}) = 1 - \sup_{x, y; x \leq y} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}.$$

Proposition 3.1 [4] The fuzzy relation Nec is antisymmetric and transitive (i.e. it is a fuzzy partial order on $\mathbb{F}(\mathbb{R})$). We use NSD index because of its property and natural interpretation employed in some problems of statistics (for more details, see [7, 16]).

Definition 3.2 Consider the problem of a non-parametric linear rank tests based on combined imprecise observations $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ and $\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_m$ (denoted by $\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_N$) whose α -cuts are $\tilde{z}_i[\alpha] = [(\tilde{z}_i)_\alpha^L, (\tilde{z}_i)_\alpha^U]$. The fuzzy linear rank test statistic is defined by a fuzzy set with the following membership function

$$\mu_{\tilde{T}_N}(w) = \sup_{\alpha \in [0, 1]} \alpha I_{\{\min_{\beta \geq \alpha} g(\beta), \dots, \max_{\beta \geq \alpha} g(\beta)\}}(w),$$

in which,

$$g(\beta) = \sum_{i=1}^n a \left[N - \sum_{j=1}^N I(Nec(\tilde{z}_j \succ \tilde{x}_i) \geq \beta) \right].$$

Remark 3.1 If the available fuzzy observations $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ and $\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_m$ reduce to the crisp values $x_1, x_2, \dots, x_n$, and $y_1, y_2, \dots, y_m$, then the fuzzy linear rank test statistic $\tilde{T}_N$ reduces to the classical linear rank test statistic T_N (see Appendix).

3.2 Fuzzy Critical Value

In the linear rank tests for comparing two statistical populations, the usual approach is to compare the observed test statistic and relative critical value. In this section, by introducing and applying the concept of fuzzy critical value, we are going to generalize this approach for location and scale problem, for fuzzy observations. We establish the approach for location

Figure 2. The α -cut of fuzzy critical value for a given significance level $\tilde{\delta}$.

problem. First, we consider the case of testing $H_0 : \theta = 0$ against $H_1 : \theta > 0$. At a significance level of δ , the common method rejects H_0 if $T_N \leq w_\delta$, where $\mathbf{P}_{H_0}(T_N \leq w_\delta) = \max_w \mathbf{P}_{H_0}(T_N \leq w) \leq \delta$ (Figure 1).

Now, assume that the level of significance is given by a triangular fuzzy number as $\tilde{\delta} = (\epsilon_1, \delta, \epsilon_2)_T$ [17]. Hence, every value satisfying $\tilde{\delta}_\alpha^L \leq \mathbf{P}_{H_0}(T_N \leq w) \leq \tilde{\delta}_\alpha^U$ is a candidate to be a critical value with a certain degree. Since the distribution of T_N is discrete, the last inequality is equivalent to $w_{\tilde{\delta}_\alpha^L}^L \leq w \leq w_{\tilde{\delta}_\alpha^U}^U$, where $\mathbf{P}_{H_0}(T_N \leq w_{\tilde{\delta}_\alpha^L}^L) = \max_w \mathbf{P}_{H_0}(T_N \leq w) \leq \tilde{\delta}_\alpha^L$ and $\mathbf{P}_{H_0}(T_N \leq w_{\tilde{\delta}_\alpha^U}^U) = \max_w \mathbf{P}_{H_0}(T_N \leq w) \leq \tilde{\delta}_\alpha^U$ (Figure 2). Based on Appendix, for large sample sizes N , $\mathbf{P}_{H_0}(\cdot)$ can be found by normal approximation. Hence, we can define the fuzzy critical as follows.

Definition 3.3 Consider the problem of location hypothesis tests $H_0 : \theta = 0$ and $H_1 : \theta > 0$. At the fuzzy significance level $\tilde{\delta} = (\epsilon_1, \delta, \epsilon_2)_T$, the fuzzy critical value is defined to be a fuzzy set with the following membership function

$$\mu_{(\tilde{R}_{T_N}^c)}(y) = \sup_{\alpha \in [0,1]} \alpha I_{\{w_{\tilde{\delta}_\alpha^L}^L, \dots, w_{\tilde{\delta}_\alpha^U}^U\}}(y),$$

in which

$$w_{\tilde{\delta}_\alpha^L}^L = \max\{w : \mathbf{P}_{H_0}(T_N \leq w) \leq \tilde{\delta}_\alpha^L\},$$

$$w_{\tilde{\delta}_\alpha^U}^U = \max\{w : \mathbf{P}_{H_0}(T_N \leq w) \leq \tilde{\delta}_\alpha^U\}.$$

Definition 3.4 Suppose we wish to test $H_0 : \theta = 0$ against $H_1 : \theta < 0$, at the fuzzy significance level $\tilde{\delta} = (\epsilon_1, \delta, \epsilon_2)_T$. The fuzzy critical value is defined to be a fuzzy set with a membership function as follows

$$\mu_{(\tilde{R}_{T_N}^c)}(y) = \sup_{\alpha \in [0,1]} \alpha I_{\{w_{\tilde{\delta}_\alpha^L}^L, \dots, w_{\tilde{\delta}_\alpha^U}^U\}}(y),$$

where

$$w_{\tilde{\delta}_\alpha^L}^L = \min\{w : \mathbf{P}_{H_0}(T_N \geq w) \leq \tilde{\delta}_\alpha^U\},$$

$$w_{\tilde{\delta}_\alpha^U}^U = \min\{w : \mathbf{P}_{H_0}(T_N \geq w) \leq \tilde{\delta}_\alpha^L\}.$$

Now, we consider the case of testing $H_0 : \theta = 0$ against the alternative $H_1 : \theta \neq 0$. In the classical case, at a crisp significance level of δ , we reject the null hypothesis H_0 when $T_N \leq w_\delta$ or $T_N \geq w'_\delta$, where w_δ and w'_δ are the critical values such that $\mathbf{P}_{H_0}(T_N \leq w_\delta) = \max_w \mathbf{P}_{H_0}(T_N \leq w) \leq \delta/2$ and $\mathbf{P}_{H_0}(T_N \geq w'_\delta) = \min_w \mathbf{P}_{H_0}(T_N \geq w) \leq \delta/2$. Now, if the level of significance is a triangular fuzzy number as $\tilde{\delta} = (\epsilon_1, \delta, \epsilon_2)_T$, then, using a similar way to that of one-sided cases, we can define the lower and upper fuzzy critical values as follows.

Definition 3.5 In the problem of hypothesis test $H_0 : \theta = 0$ against $H_1 : \theta \neq 0$, at the fuzzy significance level $\tilde{\delta} = (\epsilon_1, \delta, \epsilon_2)_T$, the lower and upper fuzzy critical values are defined to be fuzzy sets with the following membership functions

Fuzzy lower critical value:

$$\mu_{(\tilde{R}_{T_N}^{cL})}(y) = \sup_{\alpha \in [0,1]} \alpha I_{\{w_{\tilde{\delta}_\alpha^L}^L, \dots, w_{\tilde{\delta}_\alpha^U}^U\}}(y),$$

where,

$$w_{\tilde{\delta}_\alpha^L}^L = \max\{w : \mathbf{P}_{H_0}(T_N \leq w) \leq \tilde{\delta}_\alpha^L/2\},$$

$$w_{\tilde{\delta}_\alpha^U}^U = \max\{w : \mathbf{P}_{H_0}(T_N \leq w) \leq \tilde{\delta}_\alpha^U/2\}.$$

Fuzzy upper critical value:

$$\mu_{(\tilde{R}_{T_N}^{cU})}(y) = \sup_{\alpha \in [0,1]} \alpha I_{\{w_{\tilde{\delta}_\alpha^L}^L, \dots, w_{\tilde{\delta}_\alpha^U}^U\}}(y),$$

where,

$$w_{\tilde{\delta}_\alpha^L}^L = \min\{w : \mathbf{P}_{H_0}(T_N \geq w) \leq \tilde{\delta}_\alpha^U/2\},$$

$$w_{\tilde{\delta}_\alpha^U}^U = \min\{w : \mathbf{P}_{H_0}(T_N \geq w) \leq \tilde{\delta}_\alpha^L/2\}.$$

Using normal approximation, we can also define the fuzzy critical values, for large sample sizes, in a similar way.

Remark 3.2 Let the probability distribution of T_N be symmetric. Then the upper fuzzy critical value for two-sided alternative hypothesis test $H_1 : \theta \neq 0$, reduces as follows

$$\mu_{(\tilde{R}_{T_N}^{cU})}(y) = \sup_{\alpha \in [0,1]} \alpha I_{\{2E(T_N) - w_{\tilde{\delta}_\alpha^U}^U, \dots, 2E(T_N) - w_{\tilde{\delta}_\alpha^L}^L\}}(y).$$

Hence, based on interval arithmetic, it is easy to see that $\tilde{R}_{T_N}^{cU} =$

Table 1. Rejection regions

Alternative	Rejection region
(1) $H_1 : \theta < 0$	$\Diamond[\tilde{T}_N] \succeq \Diamond[\tilde{R}_{\tilde{T}_N}^c]$
(2) $H_1 : \theta > 0$	$\Diamond[\tilde{T}_N] \preceq \Diamond[\tilde{R}_{\tilde{T}_N}^c]$
(3) $H_1 : \theta \neq 0$	$\Diamond[\tilde{T}_N] \succeq \Diamond[\tilde{R}_{\tilde{T}_N}^{cU}] \text{ or } \Diamond[\tilde{T}_N] \preceq \Diamond[\tilde{R}_{\tilde{T}_N}^{cL}]$

$2E(T_N) \odot \tilde{R}_{\tilde{T}_N}^{cL}$, where $\odot$ denotes the generalized mines operator (for more details see [?]).

4. Method of Decision Making

Consider the problem of linear rank test for location problem with imprecise observations at a given fuzzy significance level. One can expect that, for the case of testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, if the observed fuzzy test statistic is less than the corresponding fuzzy critical value, then H_0 is rejected, otherwise H_0 is accepted (the similar argument can be stated for other two cases). To do this, we need a criterion for comparing the observed fuzzy test statistic and fuzzy critical value. The generalized rejection regions may be represented as given in Table 1, in which $\Diamond$ denotes a suitable ranking operator (see, for example, [18, 19]). One of the most commonly used methods for ranking fuzzy sets consists in the definition of the preference degree P_c [19].

Definition 4.1 For two discrete fuzzy sets $\tilde{A}$ and $\tilde{B}$ with the supports $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_m\}$, the P_c -index is defined as $P_c(\tilde{A}, \tilde{B}) = S(\tilde{A} \succ \tilde{B}) + \frac{1}{2}S(\tilde{A} \approx \tilde{B})$, where, two preference functions $S(\tilde{A} \succ \tilde{B})$ and $S(\tilde{A} \approx \tilde{B})$ are defined as follows

$$S(\tilde{A} \succ \tilde{B}) = \frac{\sum_{x=a_1}^{a_n} \sum_{y=b_1, y < x}^{b_m} \mu_{\tilde{A}}(x) \odot \mu_{\tilde{B}}(y)}{\sum_{x=a_1}^{a_n} \sum_{y=b_1}^{b_m} \mu_{\tilde{A}}(x) \odot \mu_{\tilde{B}}(y)},$$

and

$$S(\tilde{A} \approx \tilde{B}) = \frac{\sum_{x=a_1}^{a_n} \sum_{y=b_1, y=x}^{b_m} \mu_{\tilde{A}}(x) \odot \mu_{\tilde{B}}(y)}{\sum_{x=a_1}^{a_n} \sum_{y=b_1}^{b_m} \mu_{\tilde{A}}(x) \odot \mu_{\tilde{B}}(y)},$$

where $\odot$ is a t-norm operator that for $a > 0$ and $b > 0$ satisfies the condition $a \odot b > 0$ (see [14, 18]). In the following, we utilize the min-operator as the t-norm operator.

Remark 4.1 The preference degree P_c takes its values in the interval $[0, 1]$. In addition, $P_c(\tilde{A}, \tilde{B}) = 1 - P_c(\tilde{B}, \tilde{A})$. It is obvious that if $P_c(\tilde{A}, \tilde{B}) = 1$, then $\tilde{A}$ is absolutely preferred to $\tilde{B}$, if $P_c(\tilde{A}, \tilde{B}) = 0.5$ then $\tilde{A}$ and $\tilde{B}$ are equally preferred, and $\tilde{B}$ is absolutely preferred to $\tilde{A}$ if $P_c(\tilde{A}, \tilde{B}) = 0$.

Table 2. Data set in Example 5.1

Supplier A	Supplier B
(25, 35, 45) _T	(36, 46, 56) _T
(56, 66, 76) _T	(46, 56, 66) _T
(48, 58, 68) _T	(50, 60, 70) _T
(73, 83, 93) _T	(39, 49, 59) _T
(61, 71, 81) _T	-

Definition 4.2 Consider the null hypothesis $H_0 : \theta = 0$ against an alternative hypothesis given in Table 1 for location problem. Based on Definition 4.1,

- 1) in the one-sided case (1), we reject H_0 with preference degree $\eta = P_c(\tilde{T}_N, \tilde{R}_{\tilde{T}_N}^c)$, and accept H_0 with preference degree $1 - \eta$.
- 2) in the one-sided (2), we reject H_0 with preference degree $\eta = P_c(\tilde{R}_{\tilde{T}_N}^c, \tilde{T}_N)$, and accept H_0 with preference degree $1 - \eta$.
- 3) in the two-sided case (3), we reject H_0 with preference degree $\eta = \max\{P_c(\tilde{T}_N, \tilde{R}_{\tilde{T}_N}^{cU}), P_c(\tilde{R}_{\tilde{T}_N}^{cL}, \tilde{T}_N)\}$, and accept H_0 with preference degree $1 - \eta$.

Note that, the comparison approach used in pervious definition is subjective, and so nothing of the main results of the present work will be lost by altering these definitions to ones which fit the demands of the decision makers.

By similar argument as we noted in previous definition, we can define the the preference degree to accept or reject the hypothesis $H_0 : \theta = 1$ versus an alternative hypothesis given in Table 6 for scale problem.

Remark 4.2 It should be mentioned that, the proposed method of test (which is illustrated in Section 3) and the method of decision making are general. So that, one can use the methods based on non-symmetric fuzzy numbers (observations). Note that, there is not any limitation, in this regard, in the definitions of fuzzy test statistic and fuzzy critical value as well as in the definition of the depreference degree P_c .

5. Numerical Example

To clarify our proposed method, a numerical example is provided in this section.

Example 5.1 ([20], p. 384) Two potential suppliers of street lighting equipment, A and B, want to present their bids to a city manager. Two independent random samples of size 5 and 4 street lighting equipments were tested from each supplier

because the tests are expensive and may take considerable time to complete. Since, under some unexpected situations, we cannot measure the life lengths, precisely, we can just obtain the tire life around a number. The life lengths are reported to be triangular fuzzy numbers as shown in Table 2. We wish to test whether the life length of suppliers A and B have equal variability (i.e. $H_0 : \sigma_A = \sigma_B$).

Before we test for scale, we must determine whether we can assume the locations (medians) can be regarded as equal (i.e. $H_0 : M_A = M_B$). One of the most commonly used tests for the location problem is the Wilcoxon test. Using Definition 3.2, the fuzzy wilcoxon test statistic is obtained as follows

$$\widetilde{W}_N = \left\{ \frac{1}{29}, \frac{0.85}{28}, \frac{0.75}{27}, \frac{0.69}{26}, \frac{0.60}{25}, \frac{0.55}{23}, \frac{0.50}{22}, \frac{0.45}{21}, \frac{0.34}{20}, \frac{0.30}{19}, \frac{0.20}{18}, \frac{0.15}{16}, \frac{0.10}{15} \right\}.$$

At the fuzzy significance level $\widetilde{\delta} = (0.02, 0.05, 0.08)_T$, since the probability distribution of W_N is symmetric, from Definition 3.5 and Remark 3.2, the fuzzy lower and upper critical values are calculated as follows

$$\widetilde{R}_{W_N}^{cL} = \left\{ \frac{0.39}{10}, \frac{1}{11}, \frac{0.53}{12} \right\}, \quad \widetilde{R}_{W_N}^{cU} = \left\{ \frac{0.53}{28}, \frac{1}{29}, \frac{0.39}{30} \right\}.$$

By using Definition 4.1, we obtain $P_c(\widetilde{W}_N, \widetilde{R}_{W_N}^{cL}) = 0$ and $P_c(\widetilde{R}_{W_N}^{cU}, \widetilde{W}_N) \simeq 0.06$. From Definition 4.2 (part (c)), therefore, we conclude that there is no difference in the locations of the A and B populations with preference degree ($\simeq$) 0.94.

Now, we wish to test that whether the life length of suppliers A and B have equal variability. The fuzzy Siegel-Tukey test statistic can be obtained as

$$\widetilde{S}_N = \left\{ \frac{1}{21}, \frac{0.85}{24}, \frac{0.75}{25}, \frac{0.69}{26}, \frac{0.55}{27}, \frac{0.50}{30} \right\},$$

since the probability distribution of $\widetilde{S}_N$ is the same as that of the W_N , Therefore, the fuzzy critical values are remaind to that of Wilcoxon test. We obtain $P_c(\widetilde{T}_N, \widetilde{R}_{W_N}^{cL}) = 0$ and $P_c(\widetilde{R}_{W_N}^{cU}, \widetilde{T}_N) \simeq 0.07$. Therefore, we conclude that, with preference degree ($\simeq$) 0.93, there is no basis for difference between variability suppliers A and B.

6. Conclusions

As natural generalizations of the some statistical non-parametric tests, we proposed the corresponding tests based on fuzzy ob-

servations, when the given significance level is a fuzzy number, too. To do this, the usual concepts of the test statistic and critical value were extended to the concepts of the fuzzy test statistic and fuzzy critical value. For decision making (about rejection/acceptance the null hypothesis of interest) a method was used to compare the observed fuzzy test statistic and associated fuzzy critical value on the basis of a preference degree for ranking fuzzy sets. The proposed method is general and so, it can be applied for other non-parametric tests such as Mann-Whitney test and Kendall test.

Finally, it should be mentioned that, in many real-world problems, we initially come across with the fuzzy (vague) data, so that the proposed method in this article (generally the methods for fuzzy data) can be applied for such problems. Beside, we may consider the problem of testing fuzzy rather than crisp hypotheses, in which, by using crisp or fuzzy data, we wish to test some vague claims about the underlying statistical population(s) (see e.g., [21, 22]). Such extensions are suggested to expand the idea proposed in this article.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

APPENDIX A:

A Brief Review of Statistical Linear Rank Tests

Suppose that two independent random samples $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$ are drawn from populations with the continuous cumulative distribution functions F_X and F_Y , respectively. Many statistical procedures applicable to the two-sample problem are based on the rank-sum statistics for the combined samples of size $N = n + m$. These tests can be classified as linear combinations of certain indicator variables for the combined ordered samples $Z = (Z_1, Z_2, \dots, Z_N)$, where $Z_j = 1$ if the j th random variable in the combined ordered sample is an X and $Z_j = 0$ if it is a Y , for $j = 1, 2, \dots, N$. Such functions are often called linear rank statistics defined as follows

$$T_N = \sum_{j=1}^N a[j]Z_j,$$

where the $a[j]$ are given constants called weights or scores. Some properties of T_N are given as follows [20].

Table 3. Some well-known test statistics for two-sample location problem

Test statistic	$a[j]$
Wilcoxon: W_N	j
Van-Der Vaerden: V_N	$\Phi^{-1}(\frac{j}{N+1})$
Terry-Hoeffding: T_N	$E(\zeta_j)$

Under the null hypothesis H_0 , we have for all $j = 1, 2, \dots, N$,

$$E(T_N) = \frac{m}{N} \sum_{j=1}^N a[j],$$

$$Var(T_N) = \frac{mn}{N^2(N-1)} (N \sum_{j=1}^N a^2[j] - (\sum_{j=1}^N a[j])^2).$$

The null distribution of T_N is symmetric about its mean if $a[j] = a[N-j+1]$, $j = 1, 2, \dots, N$ (specially when $m = n = N/2$). The probability distribution of a standardized linear rank statistic $\frac{T_N - E(T_N)}{\sqrt{Var(T_N)}}$ approaches the standard normal probability distribution subject to certain regularity conditions.

Suppose that two independent samples of sizes m and n are drawn from two continuous populations. We wish to test the null hypothesis of identical distributions. For alternative hypothesis, There are two cases given as follows [20].

A.1. Location Problem

The alternative hypothesis is that the populations are of the same form but with a different measure of central tendency. This can be expressed symbolically as follows

$$\begin{cases} H_0 : F_Y(x) = F_X(x) \\ H_1 : F_Y(x) = F_X(x - \theta), \quad \theta \neq 0. \end{cases}$$

Note that, the random variable Y is stochastically larger than X when $\theta > 0$, and Y is stochastically smaller than X when $\theta < 0$. Thus, for example, when $\theta < 0$, the median of X (M_X) is larger than the median of Y (M_Y).

Some well-known linear rank statistics used in two-sample location problem are given in Table 3. In the table $\Phi(x)$ and ζ_i denote the cumulative standard normal distribution and the j th-order statistic from a standard normal population, respectively.

Table 4 shows the appropriate rejection regions for general two-sample location problem in terms of difference medians of two unknown populations F_Y and F_X at a given δ level significance. The critical values can be easily found from the related tables provided for example in [20].

Table 4. Rejection regions for general two-sample location problem

Alternative hypothesis: H_0	Rejection region
$\theta < 0$	$T_N \geq w'_\delta$
$\theta > 0$	$T_N \leq w_\delta$
$\theta \neq 0$	$T_N \leq w_{\delta/2}$ or $T_N \geq w'_{\delta/2}$

Table 5. Some well-known test statistics for two-sample scale problem

Test Statistic	$a[j]$
Mood: M_N	$(j - (N+1)/2)^2$
Ansari-Bradley: A_N	$ j - (N+1)/2 $
Klotz Normal-Scores: K_N	$(\Phi^{-1}(j/(N+1)))^2$
Siegel-Tukey: S_N	$\begin{cases} 2j & j \text{ even}, 1 \leq j \leq N/2, \\ 2j-1 & j \text{ odd}, 1 \leq j \leq N/2, \\ 2(N-j)+2 & j \text{ even}, N/2 < j \leq N, \\ 2(N-j)+1 & j \text{ odd}, N/2 < j \leq N. \end{cases}$

Note that if the distribution of T_N is symmetric then $w'_{\delta/2} = 2E(T_N) - w_{\delta/2}$. In addition, When m and n are too large, to find related critical value(s), we can use normal approximations. For example, the Wilcoxon's large sample test statistic is given by

$$Z = \frac{W_N - \frac{mn}{2}}{\sqrt{\frac{mn(N+1)}{12}}}.$$

whose distribution is approximately standard normal.

A.2. Scale Problem

Now, we are interested in detecting differences in variability between two populations. This can be expressed symbolically as follows

$$\begin{cases} H_0 : F_{Y-M}(x) = F_{X-M}(x) \\ H_1 : F_{Y-M}(x) = F_{X-M}(\theta x), \quad \theta > 0, \theta \neq 1, \end{cases}$$

where M is interpreted to be the common median and $\theta = \sigma_X/\sigma_Y$. The alternative hypothesis H_1 appropriately called the scale alternative because the cumulative distribution function of the Y population is the same as that of the X population but with a compressed or enlarged scale according as $\theta > 1$ or $\theta < 1$, respectively.

Some well-known linear rank statistics used in two-sample scale problem are given in Table 5.

In Table 5, $\Phi(x)$ denotes the cumulative standard normal probability distribution. Table 6 shows the appropriate rejection regions for the alternative hypothesis test with a given δ level

Table 6. Rejection regions for general two-sample scale problem

Alternative hypothesis: H_1	Rejection region
$\theta > 1$	$T_N \leq w_\delta$
$\theta < 1$	$T_N \geq w'_\delta$
$\theta \neq 1$	$T_N \leq w_{\delta/2}$ or $T_N \geq w'_{\delta/2}$

significance.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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