On Common Fixed Point for Single and Set-Valued Maps Satisfying OWC Property in IFMS using Implicit Relation

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Abstract

In this paper, we introduce the notion of single and set-valued maps satisfying OWC property in IFMS using implicit relation. Also, we obtain common fixed point theorems for single and set-valued maps satisfying OWC properties in IFMS using implicit relation.

Keywords: Common fixed point, Occasionally weakly compatible map, Implicit relation.

1. Introduction

Several authors [1–5] studied and developed the various concepts in different direction and proved some fixed point in fuzzy metric space. Also, Jungck [6] introduced the concept of compatible maps, and Vijayaraju and Sajath [7] obtained some common fixed point theorems in fuzzy metric space. Recently, Park et al. [8] introduced the intuitionistic fuzzy metric space (IFMS), Park [12, 13] studied the compatible and weakly compatible maps in IFMS, and proved common fixed point theorem in IFMS. Also, Park [9] proved some properties for several types compatible maps, and Park [10] defined occasionally weakly semi-compatible map and obtained some fixed point using this maps in IFMS.

In this paper, we introduce the notion of single and set-valued maps satisfying occasionally weakly compatible (OWC) property in IFMS using implicit relation. Also, we obtain common fixed point theorems for single and set-valued maps satisfying OWC property in IFMS using implicit relation.

2. Preliminaries

In this part, we recall some definitions, properties and known results in the IFMS as follows : Let us recall ([11]) that a continuous t -norm is an operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ which satisfies the following conditions: (a) $*$ is commutative and associative, (b) $*$ is continuous, (c) $a * 1 = a$ for all $a \in [0, 1]$, (d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ ($a, b, c, d \in [0, 1]$). Also, a continuous t -conorm is an operation $\diamond$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ which satisfies the following conditions: (a) $\diamond$ is commutative and associative, (b) $\diamond$ is continuous, (c) $a \diamond 0 = a$ for all $a \in [0, 1]$, (d) $a \diamond b \geq c \diamond d$ whenever $a \leq c$ and $b \leq d$ ($a, b, c, d \in [0, 1]$).

Definition 2.1. ([8]) The 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space (IFMS) if X is an arbitrary set, $*$ is a continuous t -norm, $\diamond$ is a continuous t -conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions; for

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all x, y, z in X and all $s, t \in (0, \infty)$,

- (a) $M(x, y, t) > 0$,
- (b) $M(x, y, t) = 1$ if and only if $x = y$,
- (c) $M(x, y, t) = M(y, x, t)$,
- (d) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
- (e) $M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous,
- (f) $N(x, y, t) > 0$,
- (g) $N(x, y, t) = 0$ if and only if $x = y$,
- (h) $N(x, y, t) = N(y, x, t)$,
- (i) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$,
- (j) $N(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous.

Note that (M, N) is called an IFM on X . The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t , respectively

Through out this paper, X will represent the IFMS and $CB(X)$, the set of all non-empty closed and bounded subsets of X . For $A, B \in CB(X)$ and for every $t > 0$, denote

$$\begin{aligned} H(A, B, t) &= \sup\{M(a, b, t); a \in A, b \in B\}, \\ h(A, B, t) &= \inf\{N(a, b, t); a \in A, b \in B\}, \\ \delta_M(A, B, t) &= \inf\{M(a, b, t); a \in A, b \in B\}, \\ \delta^N(A, B, t) &= \sup\{N(a, b, t); a \in A, b \in B\}. \end{aligned}$$

If A consists of a single point a , we write

$$\delta_M(A, B, t) = \delta_M(a, B, t), \quad \delta^N(A, B, t) = \delta^N(a, B, t).$$

Furthermore, if B consists of a single point b , we write

$$\delta_M(A, B, t) = M(a, b, t), \quad \delta^N(A, B, t) = N(a, b, t).$$

It follows immediately from definition that

$$\begin{aligned} \delta_M(A, B, t) &= \delta_M(B, A, t) \geq 0, \\ \delta^N(A, B, t) &= \delta^N(B, A, t) \leq 1. \end{aligned}$$

Also, $\delta_M(A, B, t) = 1$ and $\delta^N(A, B, t) = 0$ if and only if $A = B = \{a\}$ for all $A, B \in CB(X)$.

Definition 2.2. Let X be an IFMS, $A : X \rightarrow X$ and $B : X \rightarrow CB(X)$.

- (a) A point $x \in X$ is called a coincidence point of hybrid maps A and B if $x = Ax \in Bx$.
- (b) Hybrid maps A and B are said to be compatible if $ABx \in$

$CB(X)$ for all $x \in X$ and

$$\begin{aligned} \lim_{n \rightarrow \infty} H(ABx_n, BAx_n, t) &= 1, \\ \lim_{n \rightarrow \infty} h(ABx_n, BAx_n, t) &= 0 \end{aligned}$$

whenever $\{x_n\}$ is a sequence in X such that $Bx_n \rightarrow D \in CB(X)$ and $Ax_n \rightarrow x \in D$.

(c) Hybrid maps A and B are said to be weakly compatible if $ABx = BAx$ whenever $Ax \in Bx$.

(d) Hybrid maps A and B are said to be occasionally weakly compatible (OWC) if there exists some points $x \in X$ such that $Ax \in Bx$ and $ABx \subseteq BAx$.

Example 2.3. Let $X = [0, \infty)$ with $a * b = \min\{a, b\}$, $a \diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$ and for all $t > 0$,

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \quad N(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

Define the maps $A : X \rightarrow X$ and $B : X \rightarrow CB(X)$ by

$$\begin{aligned} Ax &= \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ x + 1 & \text{if } 1 \leq x < \infty, \end{cases} \\ Bx &= \begin{cases} \{0\} & \text{if } 0 \leq x < 1, \\ [1, x + 3] & \text{if } 1 \leq x < \infty. \end{cases} \end{aligned}$$

Here 1 is a coincidence point of A and B , but A and B are not weakly compatible as $BA(1) = [1, 5] \neq AB(1) = [2, 5]$. Also, A and B are OWC hybrid maps as A and B are weakly compatible at $x = 0$ as $A(0) \in B(0)$ and $0 = AB(0) \subseteq BA(0) = \{0\}$. Hence weakly compatible hybrid maps are OWC, but the converse is not true in general.

3. Main Results

Theorem 3.1. Let X be an IFMS with $t * t = t$ and $t \diamond t = t$ for all $t \in [0, 1]$. Also, let $A, B : X \rightarrow X$ and $S, T : X \rightarrow CB(X)$ be single and set-valued mappings such that the hybrid pairs (A, S) and (B, T) are OWC satisfying

$$\begin{aligned} &\phi\{\delta_M(Sx, Ty, t), M(Ax, By, t), \\ &\quad H(Ax, Sx, t), H(By, Ty, t), \\ &\quad H(Ax, Ty, t) * H(By, Sx, t)\} \geq 0 \quad (1) \\ &\psi\{\delta^N(Sx, Ty, t), N(Ax, By, t), \\ &\quad h(Ax, Sx, t), h(By, Ty, t), \\ &\quad h(Ax, Ty, t) \diamond h(By, Sx, t)\} \leq 1 \end{aligned}$$

for every $x, y \in X, t > 0$.

Also, let implicit relation $\Phi = \{\phi, \psi\}$ such that $\phi : [0, 1]^5 \rightarrow [0, 1]$ and $\psi : [0, 1]^5 \rightarrow [0, 1]$ continuous functions satisfying

(a) $\phi(t_1, t_2, t_3, t_4, t_5)$ is non-increasing in t_2 and t_5 for all $t > 0$. $\psi(t_1, t_2, t_3, t_4, t_5)$ is non-decreasing in t_2 and t_5 for all $t > 0$.

(b) $\phi(t, t, 1, 1, t) \geq 0$ implies that $t = 1$, and $\psi(t, t, 0, 0, t) \leq 1$ implies that $t = 0$ for all $t > 0$.

Then A, B, S and T have a unique common fixed point in X .

Proof Since the hybrid pairs (A, S) and (B, T) are OWC maps, there exist two elements $u, v \in X$ such that $Au \in Su, ASu \subseteq SAu$ and $Bv \in Tv, BTv \subseteq TBv$.

First, we prove that $Au = Bv$. As $Au \in Su$ and $Bv \in Tv$, so,

$$\begin{aligned} M(Au, Bv, t) &\geq \delta_M(Su, Tv, t), \\ M(Au, Tv, t) &\geq \delta_M(Su, Tv, t), \\ M(Bv, Su, t) &\geq \delta_M(Su, Tv, t), \\ N(Au, Bv, t) &\leq \delta^N(Su, Tv, t), \\ N(Au, Tv, t) &\leq \delta^N(Su, Tv, t), \\ N(Bv, Su, t) &\leq \delta^N(Su, Tv, t). \end{aligned}$$

If $Au \neq Bv$, then $\delta_M(Su, Tv, t) < 1$ and $\delta^N(Su, Tv, t) > 0$. Using (1) for $x = u$ and $y = v$, we have

$$\begin{aligned} \phi\{\delta_M(Su, Tv, t), M(Au, Bv, t), 1, 1, \\ M(Au, Tv, t) * M(Su, Bv, t)\} &\geq 0 \\ \psi\{\delta^N(Su, Tv, t), N(Au, Bv, t), 0, 0, \\ N(Au, Tv, t) \diamond N(Su, Bv, t)\} &\leq 1. \end{aligned}$$

That is,

$$\begin{aligned} \phi\{\delta_M(Su, Tv, t), \delta_M(Su, Tv, t), \\ 1, 1, \delta_M(Su, Tv, t)\} &\geq 0 \\ \psi\{\delta^N(Su, Tv, t), \delta^N(Au, Bv, t), \\ 0, 0, \delta^N(Au, Tv, t)\} &\leq 1. \end{aligned}$$

Also, ϕ, ψ satisfies (b), so

$$\delta_M(Su, Tv, t) = 1 \text{ and } \delta^N(Su, Tv, t) = 0.$$

This is a contradiction which gives $Au = Bv$

Now, we prove that $A^2u = Au$. Suppose that $A^2u \neq Au$, then $\delta_M(SAu, Tv, t) < 1$ and $\delta^N(SAu, Tv, t) > 0$. Also,

using (1) for $x = Au$ and $y = v$, we get

$$\begin{aligned} \phi\{\delta_M(SAu, Tv, t), M(AAu, Bv, t), 1, 1, \\ M(AAu, Tv, t) * M(SAu, Bv, t)\} &\geq 0 \\ \psi\{\delta^N(SAu, Tv, t), N(AAu, Bv, t), 0, 0, \\ N(AAu, Tv, t) \diamond N(SAu, Bv, t)\} &\leq 1. \end{aligned}$$

Also, $Au \in Su$ and $ASu \in SAu$, so $AAu \in ASu \subseteq SAu$, $Bv \in Tv$ and $BTv \subseteq TBv$, hence

$$\begin{aligned} M(AAu, Bv, t) &\geq \delta_M(SAu, Tv, t), \\ M(Bv, SAu, t) &\geq \delta_M(SAu, Tv, t), \\ N(AAu, Bv, t) &\leq \delta^N(SAu, Tv, t), \\ N(Bv, SAu, t) &\leq \delta^N(SAu, Tv, t). \end{aligned}$$

Therefore

$$\begin{aligned} \phi\{\delta_M(SAu, Tv, t), \delta_M(SAu, Tv, t), \\ 1, 1, \delta_M(SAu, Tv, t)\} &\geq 0 \\ \psi\{\delta^N(SAu, Tv, t), \delta^N(SAu, Tv, t), \\ 0, 0, \delta^N(SAu, Tv, t)\} &\leq 1. \end{aligned}$$

But ϕ, ψ satisfies (b), so,

$$\delta_M(SAu, Tv, t) = 1 \text{ and } \delta^N(SAu, Tv, t) = 0,$$

a contradiction and hence $A^2u = Au = Bv$. Similarly, we can show that $B^2v = Bv$.

Let $Au = Bv = z$, then $Az = z = Bz, z \in Sz$ and $z \in Tz$. Therefore z is a fixed point of A, B, S and T .

Finally, we prove the uniqueness of the fixed point. Let $z \neq z_0$ be another fixed point of A, B, S and T , then by (1), we have,

$$\begin{aligned} \phi\{\delta_M(Sz, Tz_0, t), \delta_M(Az, Tz_0, t), 1, 1, \\ \delta_M(Az, Tz_0, t) * \delta_M(Sz, Tz_0, t)\} &\geq 0 \\ \psi\{\delta^N(Sz, Tz_0, t), \delta^N(Az, Tz_0, t), 0, 0, \\ \delta^N(Az, Tz_0, t) \diamond \delta^N(Sz, Tz_0, t)\} &\leq 1. \end{aligned}$$

From (b), we get

$$\delta_M(Sz, Tz_0, t) = 1, \delta^N(Sz, Tz_0, t) = 0.$$

This is a contradiction. Hence $z = z_0$. Therefore z is unique common fixed point of A, B, S and T .

Example 3.2. Let X be an IFMS in which $X = R^+$, $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$ such that for all $t > 0$,

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \quad N(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

Define the maps A, B, S and T on X by

$$\begin{aligned} Ax &= \begin{cases} 2x - 1 & \text{if } x \leq 5, \\ 2x & \text{if } x > 5, \end{cases} \\ Bx &= \begin{cases} 3 - 2x & \text{if } x \leq 1, \\ x + 1 & \text{if } 1 > x, \end{cases} \\ Sx &= \begin{cases} \{1\} & \text{if } x < 2, \\ [2x, 2x + 5] & \text{if } x \geq 2, \end{cases} \\ Tx &= \begin{cases} \{1\} & \text{if } x = 1, \\ [x, x + 2] & \text{if otherwise.} \end{cases} \end{aligned}$$

Define $\phi : [0, 1] \rightarrow [0, 1]$, $\psi : [0, 1] \rightarrow [0, 1]$ as

$$\begin{aligned} \phi(t_1, t_2, t_3, t_4, t_5) &= \min\{t_1, t_2, t_3, t_4, t_5\}, \\ \psi(t_1, t_2, t_3, t_4, t_5) &= \max\{t_1, t_2, t_3, t_4, t_5\}. \end{aligned}$$

Here the pairs (A, S) and (B, T) are OWC and the contractive condition is satisfied. Hence 1 is a unique common fixed point of A, B, S and T .

Corollary 3.3. Let X be an IFMS, $t * t = t$ and $t \diamond t = t$ for all $t \in [0, 1]$ and let $A : X \rightarrow X$ and $S, T : X \rightarrow CB(X)$ be single and set-valued mappings such that the hybrid pair (A, S) and (A, T) are OWC satisfying

$$\begin{aligned} \phi\{M(Sx, Ty, t), M(Ax, Ay, t), \\ H(Ax, Sx, t), H(Ay, Ty, t), \\ H(Ax, Sy, t) * H(Ay, Sx, t)\} &\geq 0 \\ \psi\{N(Sx, Ty, t), N(Ax, Ay, t), \\ h(Ax, Sx, t), h(Ay, Ty, t), \\ h(Ax, Sy, t) \diamond h(Ay, Sx, t)\} &\leq 1 \end{aligned}$$

for every $x, y \in X$, $t > 0$ and ϕ, ψ are satisfies (a) and (b), respectively in Theorem 3.1. Then A, S and T have a unique common fixed point in X .

Proof Suppose that $A = B$ in Eq. (1) of Theorem 3.1, then we get this corollary.

Corollary 3.4. Let X be an IFMS, $t * t = t$ and $t \diamond t = t$ for all $t \in [0, 1]$ and let $A : X \rightarrow X$ and $S : X \rightarrow CB(X)$ be single and set-valued mappings such that the hybrid pair (A, S) is OWC satisfying

$$\begin{aligned} \phi\{\delta_M(Sx, Sy, t), M(Ax, Ay, t), \\ H(Ax, Sx, t), H(Ay, Sy, t), \\ H(Ax, Sy, t) * H(Ay, Sx, t)\} &\geq 0 \\ \psi\{\delta^N(Sx, Sy, t), N(Ax, Ay, t), \\ h(Ax, Sx, t), h(Ay, Sy, t), \\ h(Ax, Sy, t) \diamond h(Ay, Sx, t)\} &\leq 1 \end{aligned}$$

for every $x, y \in X$, $t > 0$ and ϕ, ψ are functions satisfying (a) and (b), respectively in Theorem 3.1. Then A and S have a unique common fixed point in X .

Proof Suppose that $A = B$ and $S = T$ in Eq. (1) of Theorem 3.1, then we get this corollary.

4. Conclusion

Park et.al. [8] introduced the IFMS, and proved common fixed point theorem in IFMS. Also, Park [9] proved some properties for several types compatible maps, and Park [10] defined occasionally weakly semi-compatible map and obtained some fixed point using this maps in IFMS.

In this paper, we introduce the notion of single and set-valued maps satisfying OWC property in IFMS using implicit relation. Also, we obtain common fixed point theorems for single and set-valued maps satisfying OWC property in IFMS using implicit relation.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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